

Radical Affinity and Variant Proportion in Natural Numbers

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Problematic

The term ‘Radical Affinity’ in the title of this document stems from an inquiry into the properties of *natural numbers* – their tendencies to behave, according to characteristics of their radices (or ‘bases’), in ways previously unacknowledged in the analyses of quantitative systems. This inquiry addresses some concerns over conventional approaches to quantitative understanding, with respect to the definition of an ‘integer’, and to the principle of *rational proportionality* governing integers in the denotation of numeric value. The inquiry begins from an empirical comparison of values in exponential series across a limited range of diverse number radices (base-2 to base-9), in terms of the logarithmic ratios of sequential values in each exponential series, relative to the ratios of corresponding values in the decimal series. While the logarithmic ratios of sequential values in the decimal series are naturally consistent, and would produce graphs consisting of horizontal straight lines, in the case of each radical series (with a limited number of exceptions) the distributions revealed are mostly irregular series of variegated peaks and troughs, displaying proportional inconsistency.

The following datasets are intended to explore comparisons between the decimal exponential series ($10^0, [...], 10^{10}$) with its corresponding series in a range of radices from binary to nonary (base-9). In what follows I have used the term ‘ z ’ to refer to the exponential index, and the term ‘ b ’ to refer to the radical index or base. The decimal series is represented by sequential values of $s=10^z_{10}$. Values in each respective corresponding radix are represented by sequential values of $s=(x^z)_b$, i.e., for $x=10_{10}$; $z=(0, [...], 10)$; $b=(2, [...], 9)$. Generally, s is equal to $(x^z)_b$, and is employed here to represent the exponential *series* of any radix, whereas x retains association with the initial decimal value (10^1).

In the decimal series, for $z = (0, [...], 10)$, $s = (1, [...], 10000000000)$.

The following table shows distributions of values corresponding to $s=10^z_{10}$, in terms of $s=(x^z)_b$ for each of the respective radices:

z	$s = 10^z_{10}$	$s = 1010^z_2$	$s = 101^z_3$
0	1	1	1
1	10	1010	101
2	100	1100100	10201
3	1000	1111101000	1101001
4	10000	10011100010000	111201101
5	100000	11000011010100000	12002011201
6	1000000	11110100001001000000	1212210202001
7	10000000	100110001001011010000000	200211001102101
8	100000000	101111101011110000100000000	20222011112012201
9	1000000000	111011100110101100101000000000	2120200200021010001
10	10000000000	1001010100000010111100100000000000	221210220202122010101

$s = 22^z_4$	$s = 20^z_5$	$s = 14^z_6$	$s = 13^z_7$	$s = 12^z_8$	$s = 11^z_9$
1	1	1	1	1	1
22	20	14	13	12	11
1210	400	244	202	144	121
33220	13000	4344	2626	1750	1331
2130100	310000	114144	41104	23420	14641
120122200	11200000	2050544	564355	303240	162151
3310021000	224000000	33233344	11333311	3641100	1783661
212021122000	10030000000	554200144	150666343	46113200	20731371
11331132010000	201100000000	13531202544	2322662122	575360400	228145181
323212230220000	4022000000000	243121245344	33531600616	7346545000	2520607101
21110002332100000	1304400000000000	4332142412144	502544411644	112402762000	27726678111

Clearly, within the terms of each respective series, the ratio: s_n/s_{n-1} is constant for each value of z :

$$s_n/s_{n-1} = 10_{10} = (x)_b$$

and in a graphical representation with z as the horizontal axis, would produce horizontal straight lines at $y=10_{10}$, and $y=(x)_b$.

However, for the non-decimal series, if we calculate s_n/s_{n-1} dividing the figures according to base-10 rules (i.e., treating them *as if* they were decimal values) instead of base- b rules, in each case the resulting series becomes inconsistent above a certain (variable) value of z .

The following tables display the resulting distributions of values of $(s_n/s_{n-1})_{10}$ for each series $s=(x^z)_b$ (the expression ' $(s_n/s_{n-1})_{10}$ ' is employed here simply to imply that the sequential radical values of $(x^z)_b$ are divided *as if* they were decimal values):

z	$(s_n/s_{n-1})_{10}$			
	$[s = 1010^z_2]$	$[s = 101^z_3]$	$[s = 22^z_4]$	$[s = 20^z_5]$
0	-	-	-	-
1	1010	101	22	20
2	1089.207920792	101	55	20
3	1010	107.930693069	27.454545455	32.5
4	9010.072000655	101	64.121011439	23.846153846
5	1098.781452499	107.930686774	56.392751514	36.129032258
6	1010.008080065	101.000589126	27.555447702	20
7	9010.720064805	165.161950272	64.054313251	44.776785714
8	1010.000000001	101.003496315	53.443411218	20.049850449
9	1097.912088781	104.846159379	28.524266590	20
10	9017.207279337	104.334590762	65.313129759	32.431626057

$(S_n/S_{n-1})_{10}$	$[s = 14^z]_6$	$[s = 13^z]_7$	$[s = 12^z]_8$	$[s = 11^z]_9$
z				
0	-	-	-	-
1	14	13	12	11
2	17.428571429	15.538461538	12	11
3	17.803278689	13	12.152777778	11
4	26.276243094	15.652703732	13.382857143	11
5	17.964536025	13.729928961	12.947907771	11.075131480
6	16.207086510	20.081882857	12.007320934	11
7	16.676027065	13.294115285	12.664634314	11.622932272
8	24.415732638	15.415932157	12.477130193	11.004828431
9	17.967452971	14.436710488	12.768596866	11.048259227
10	17.818855798	14.987188276	15.300084870	11

If we examine the ratio S_n/S_{n-1} logarithmically, we can more simply employ subtraction rather than division in determining the series.

Generally, $\log_b x$ is given by: $\log_{10}x/\log_{10}b$.

As it is conventional to derive radical logarithms from decimal logarithms, we may do so for the values of $S=(x^z)_b$ given in the first table of values on pages 2-3 above, which allows us to express the ratio S_n/S_{n-1} in terms of:

$$r = (\log_b S_n) - (\log_b S_{n-1}).$$

The following pages 5-12 display tables showing the values for $\log_b S$ [i.e., $\log_b(x^z)_b$] and r for each radical data series binary to nonary given in the first table on pages 2-3 above, i.e., for the initial decimal value of $x=10$. Graphical representations of the tabular data in terms of r against z are displayed as vertical and horizontal axes respectively in the following graphs (n.b. the vertical axes in these graphs are not at a constant scale).*

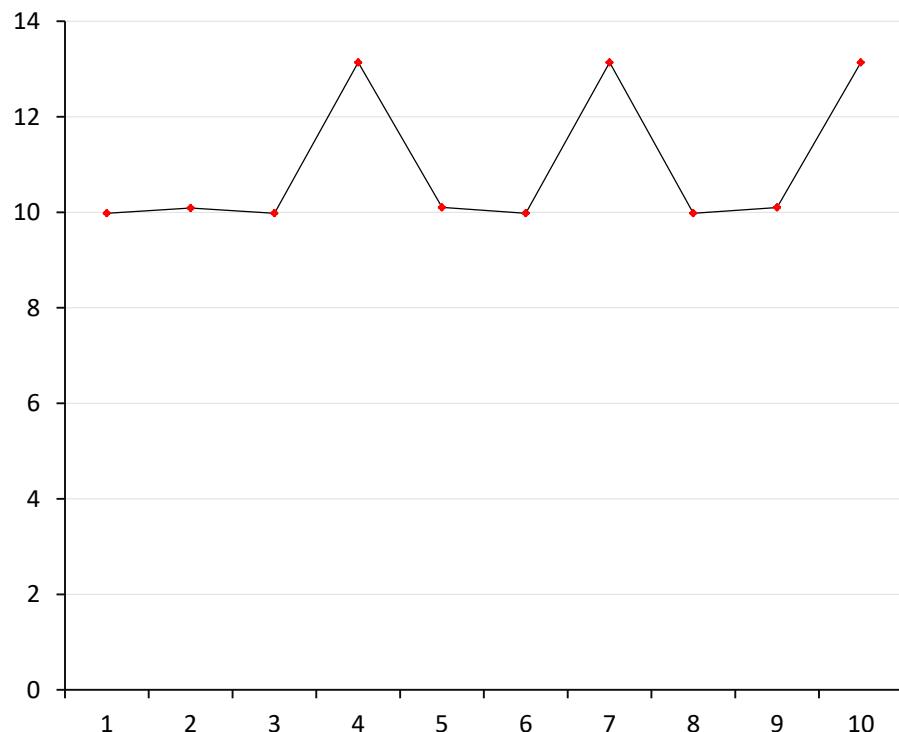
* Some may find it surprising, or erroneous, that the values of z in the horizontal axes in the following graphs are not aligned with the divisional markers, but between these points. I must confess to being a novice at the use of the 'chart' function in Microsoft Excel, and so did not override this default configuration when initially entering the data, which is the configuration most suitable when creating bar charts, for instance, rather than linear distributions of precise values. The subsequent decision not to override the default configuration for these graphs, resulting in an unconventional display, was made with regard to the fact that the precise alignment of integers with divisional markers is characteristic only of a certain limited definition of integers, as discrete *points* of value on a linear scale. With consideration to the scope of this investigation, and its limitation to the sphere of *natural numbers* – as 'wholes' (excluding fractions) – the resulting unaligned scale of values helps to accommodate certain roles of natural numbers in describing the (not strictly linear) apportionment of numerical value corresponding to familiar divisions in space and time, for instance. In identifying certain periods of time, or regions in space, we are accustomed to using whole numbers to represent entire periods or regions (the word 'zone' covers both uses). We may speak of 'Week 1' to cover any point in time in a particular 7-day duration, for instance; or use a numeric description for concentric zones in space, such as in the London Underground map, where 'Zone 2' describes *any point between* the lines of concentric division separating it from zones 1 & 3 (stations occurring precisely on the line are understood to occupy both adjacent zones simultaneously). In physics, for instance, we might wish to consider a continuous sine wave in terms of its discrete single iterations, and then identify them consecutively as iterations 1, 2, & 3, etc., and so to dissociate the numeric notation from any particular point of amplitude of the sine wave (such a dissociation being impossible for any precise positional point on the horizontal axis). By allowing attention to such 'periodic' or 'zonal' features in certain ways of using integers we may perhaps help illuminate and further explain their apparent proportional inconsistencies.

$$x = 10$$

Binary

z	$s = 1010^z_2$	$\log_2 s$	r
0	1	0	-
1	1010	9.980139578	9.980139578
2	1100100	20.069203241	10.089063663
3	1111101000	30.049342819	9.980139578
4	10011100010000	43.186665738	13.137322919
5	11000011010100000	53.288354486	10.101688748
6	11110100001001000000	63.268505605	9.980151119
7	100110001001011010000000	76.405932289	13.137426684
8	1011110101110000100000000	86.386071867	9.980139578
9	11101110011010110010100000000	96.486618692	10.100546825
10	100101010000001011110010000000000	109.625083660	13.138464968

$$\log_{10} 2 = 0.301029995664$$

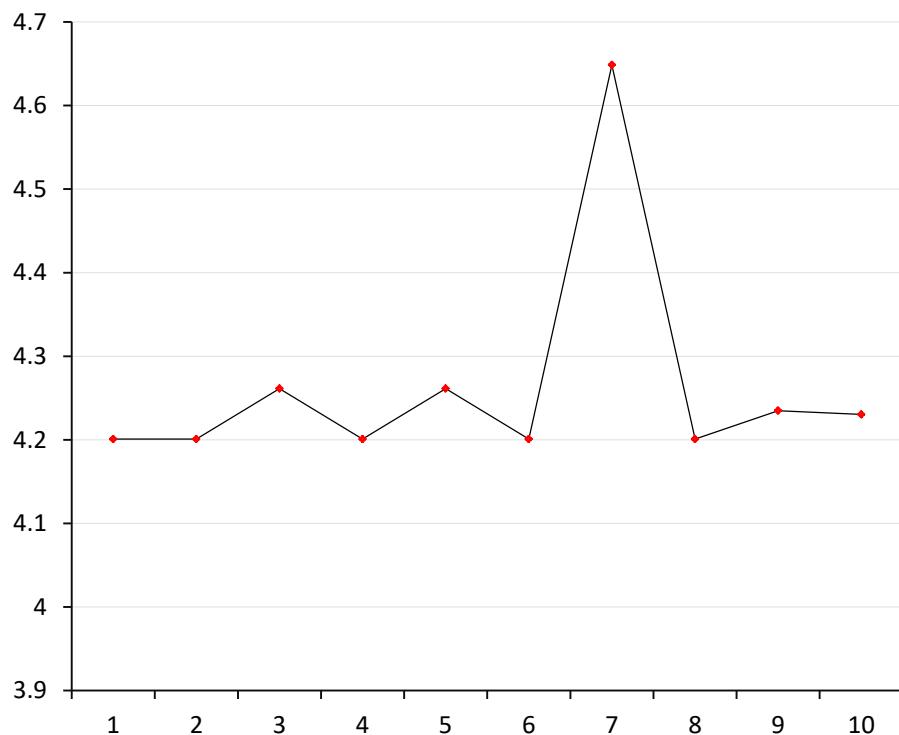


$$r = (\log_2 x^z) - (\log_2 x^{z-1}), \text{ for } x = 1010_2$$

Ternary

$$\log_{10} 3 = 0.4771212547197$$

z	$s = 101^z_3$	$\log_3 s$	r
0	1	0	-
1	101	4.200863730	4.200863730
2	10201	8.401727460	4.200863730
3	1101001	12.663002651	4.261275191
4	111201101	16.863866381	4.200863730
5	12002011201	21.125141519	4.261275138
6	1212210202001	25.326010559	4.200869040
7	200211001102101	29.974535395	4.648524836
8	20222011112012201	34.175430634	4.200895239
9	2120200200021010001	38.410313290	4.234882656
10	221210220202122010101	42.640743808	4.230430518

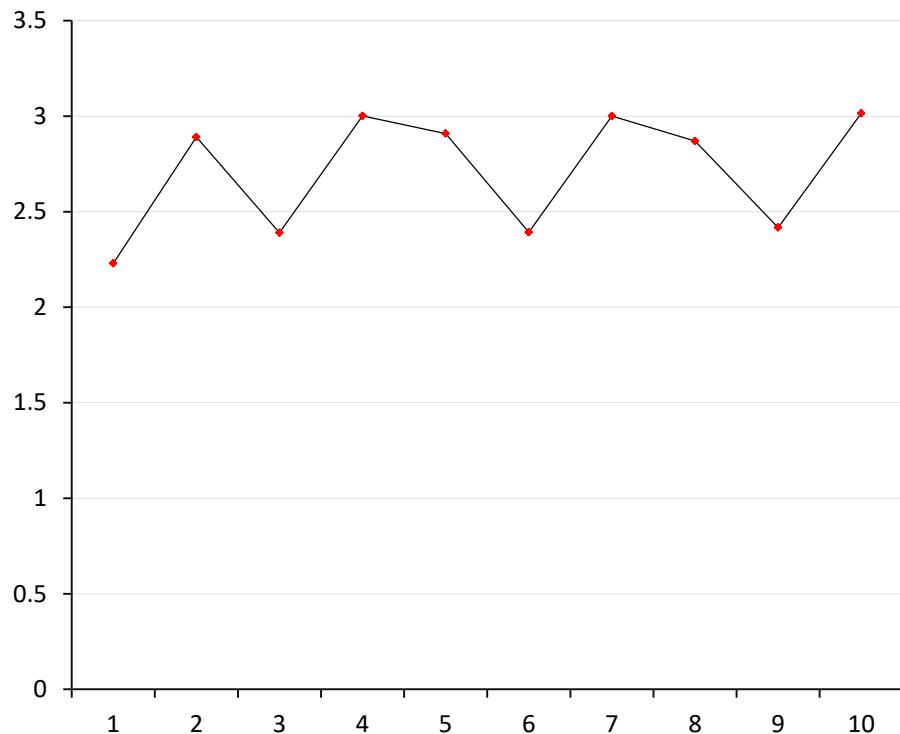


$$r = (\log_3 x^z) - (\log_3 x^{z-1}), \text{ for } x = 101_3$$

Quaternary

$$\log_{10} 4 = 0.602059991328$$

z	$s = 22_4^z$	$\log_4 s$	r
0	1	0	-
1	22	2.229715809	2.229715809
2	1210	5.120395666	2.890679857
3	33220	7.509882226	2.389486560
4	2130100	10.511244865	3.001362639
5	120122200	13.419963781	2.908718916
6	3310021000	15.812096612	2.392132831
7	212021122000	18.812708520	3.000611908
8	11331132010000	21.682678615	2.869970095
9	323212230220000	24.099737559	2.417058944
10	21110002332100000	27.114388128	3.014650569

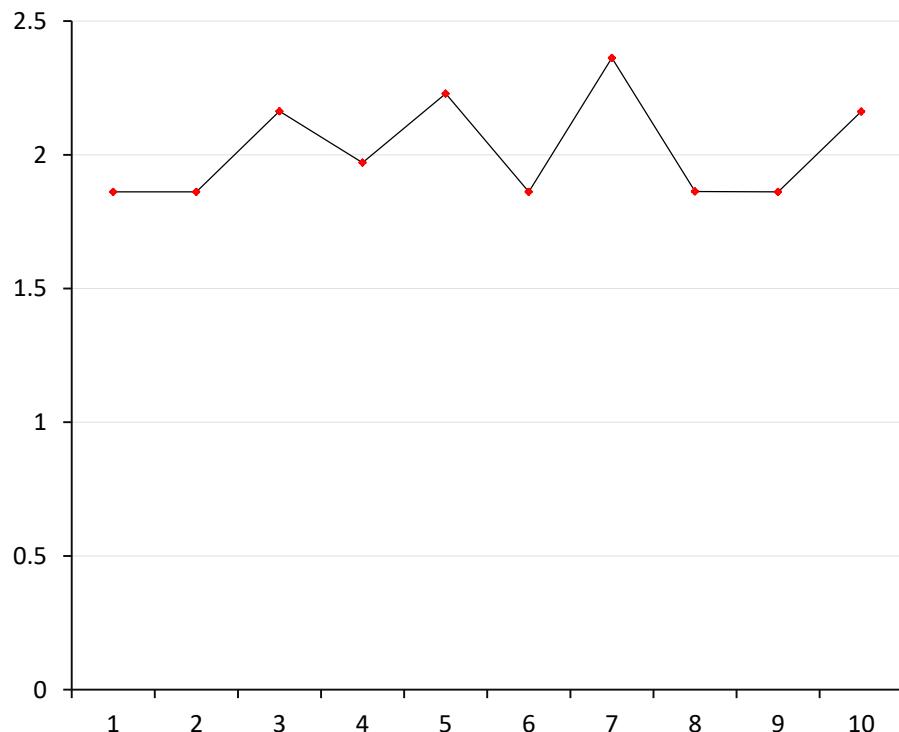


$$r = (\log_4 x^z) - (\log_4 x^{z-1}), \text{ for } x = 22_4$$

Quinary

$$\log_{10} 5 = 0.698970004336$$

z	$s = 20^z_5$	$\log_5 s$	r
0	1	0	-
1	20	1.861353116	1.861353116
2	400	3.722706232	1.861353116
3	13000	5.885722315	2.163016083
4	310000	7.856362447	1.970640132
5	11200000	10.085150978	2.228788531
6	224000000	11.946504094	1.861353116
7	10030000000	14.308626795	2.362122701
8	201100000000	16.171526676	1.862899881
9	4022000000000	18.032879792	1.861353116
10	130440000000000	20.194587325	2.161707533

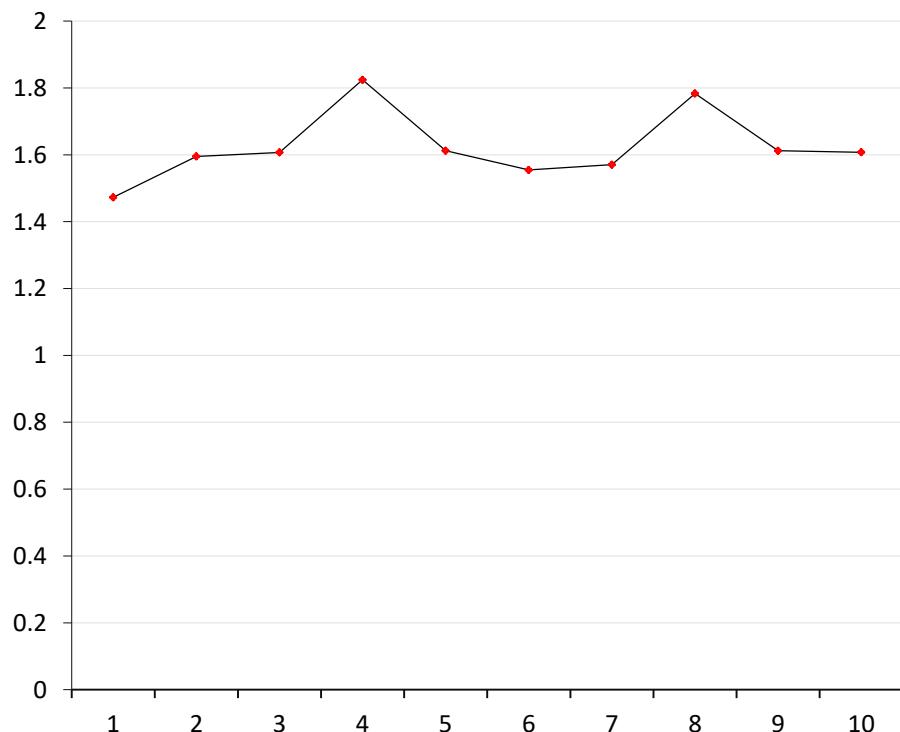


$$r = (\log_5 x^z) - (\log_5 x^{z-1}), \text{ for } x = 20_5$$

Senary

$$\log_{10} 6 = 0.7781512503836$$

z	$s = 14_6^z$	$\log_6 s$	r
0	1	0	-
1	14	1.472885940	1.472885940
2	244	3.068028002	1.595142062
3	4344	4.675042050	1.607014048
4	114144	6.499318847	1.824276797
5	2050544	8.111365354	1.612046507
6	33233344	9.665953809	1.554588455
7	554200144	11.236461588	1.570507779
8	13531202544	13.019752124	1.783290536
9	243121245344	14.631889245	1.612137121
10	4332142412144	16.239391402	1.607502157

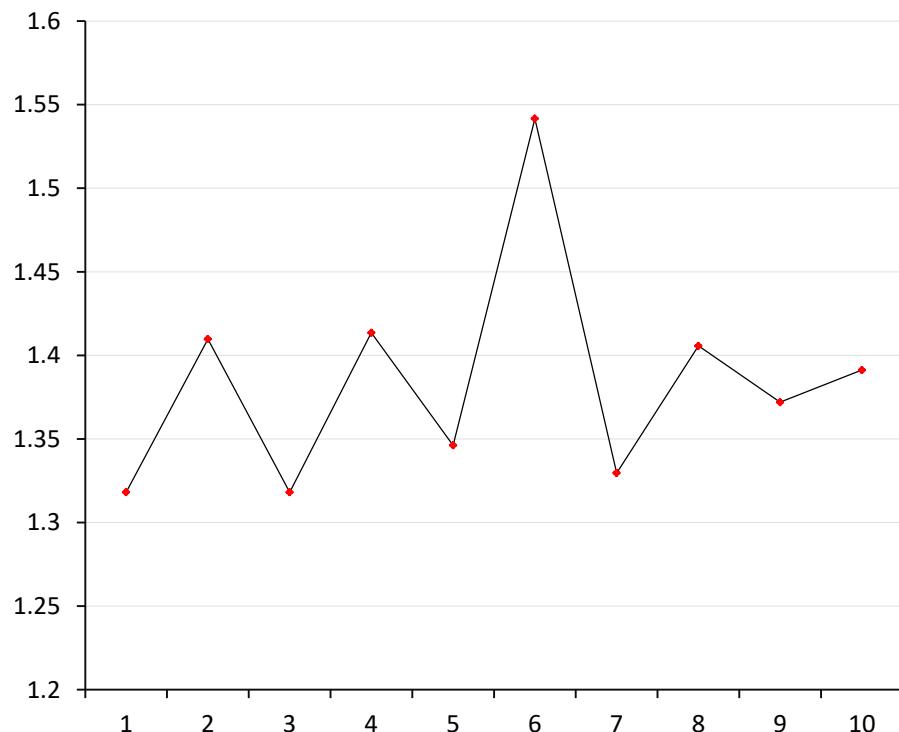


$$r = (\log_6 x^z) - (\log_6 x^{z-1}), \text{ for } x = 14_6$$

Septenary

$$\log_{10} 7 = 0.8450980400143$$

z	$s = 13^z_7$	$\log_7 s$	r
0	1	0	-
1	13	1.318123223	1.318123223
2	202	2.727909971	1.409786748
3	2626	4.046033194	1.318123223
4	41104	5.459584413	1.413551219
5	564355	6.805781229	1.346196816
6	11333311	8.347382756	1.541601527
7	150666343	9.677002975	1.329620219
8	2322662122	11.082721287	1.405718312
9	33531600616	12.454713875	1.371992588
10	502544411644	13.845937269	1.391223394

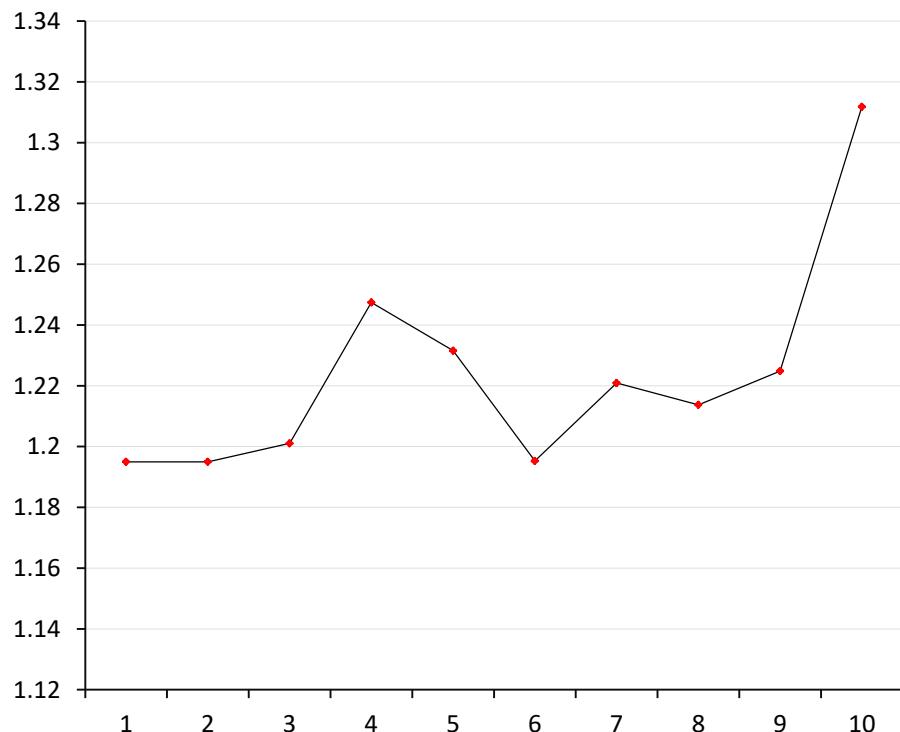


$$r = (\log_7 x^z) - (\log_7 x^{z-1}), \text{ for } x = 13_7$$

Octal

$$\log_{10} 8 = 0.9030899869919$$

z	$s = 12^z_8$	$\log_8 s$	r
0	1	0	-
1	12	1.194987500	1.194987500
2	144	2.389975000	1.194987500
3	1750	3.591046402	1.201071402
4	23420	4.838484485	1.247438083
5	303240	6.070033515	1.231549030
6	3641100	7.265314311	1.195280796
7	46113200	8.486225483	1.220911172
8	575360400	9.699963563	1.213738080
9	7346545000	10.924806260	1.224842697
10	112402762000	12.236628843	1.311822583

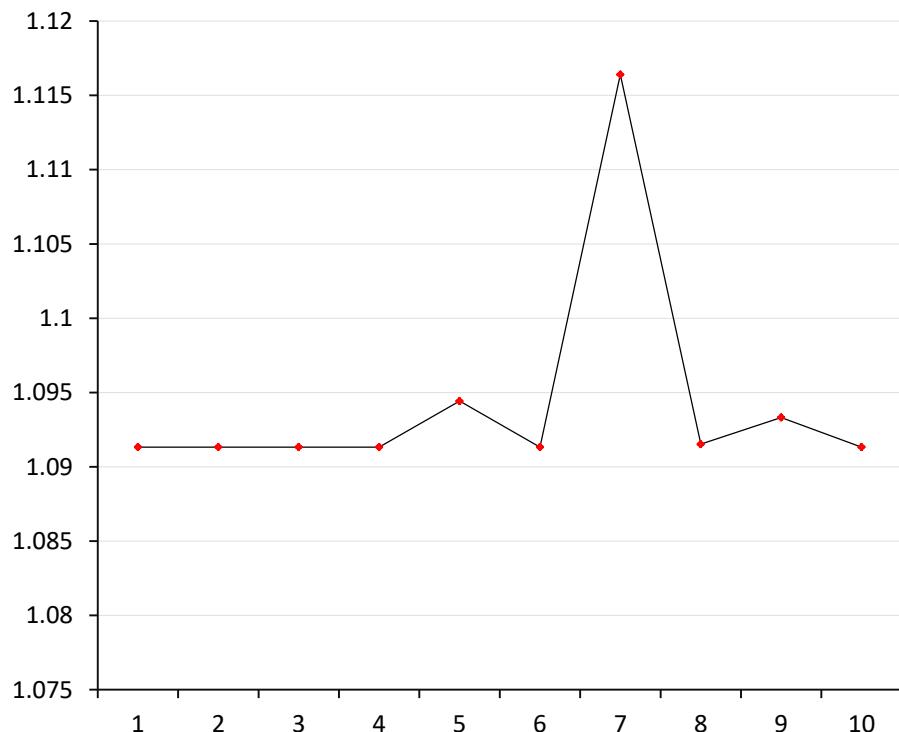


$$r = (\log_8 x^z) - (\log_8 x^{z-1}), \text{ for } x = 12_8$$

Nonary

$$\log_{10} 9 = 0.9542425094393$$

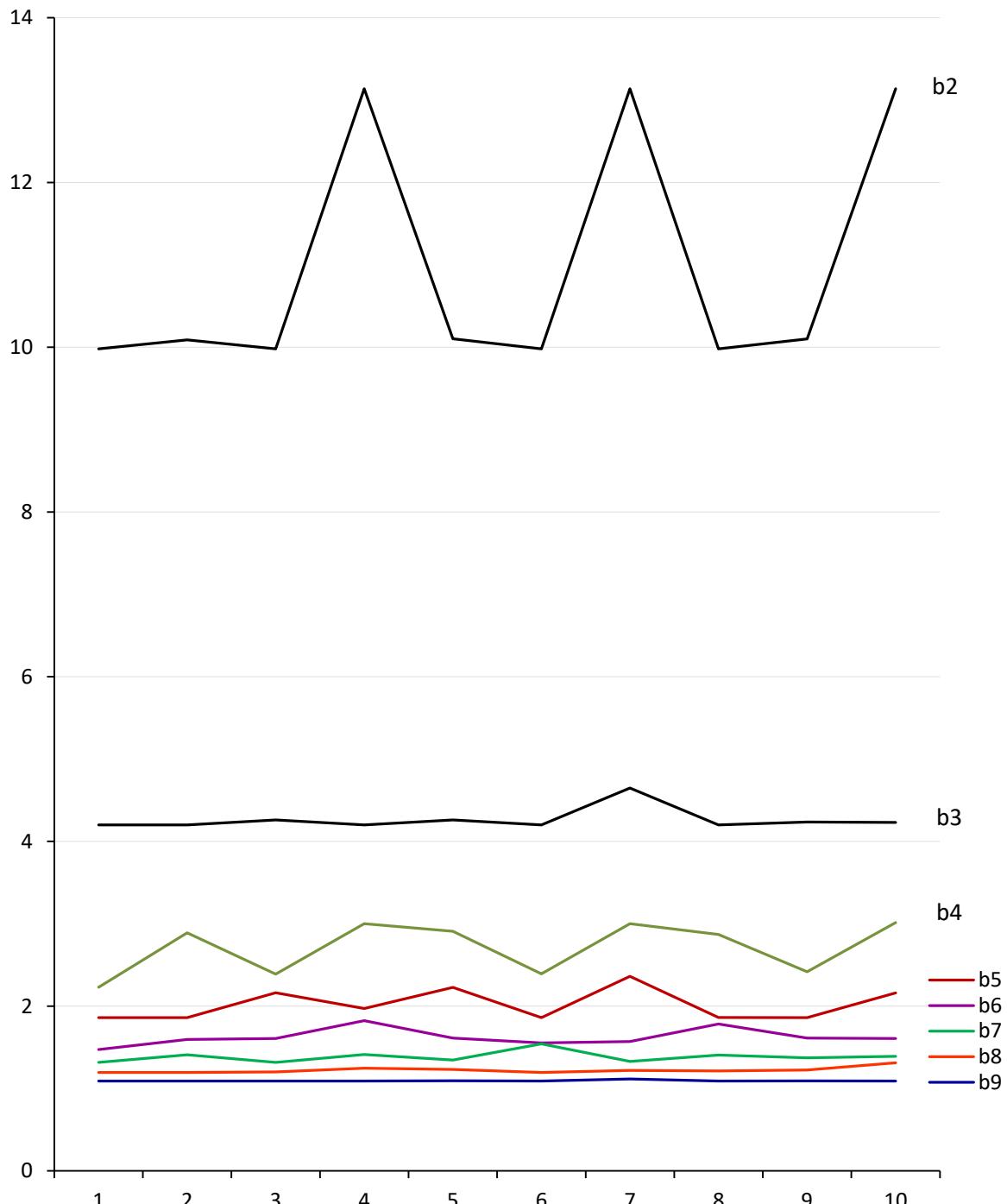
z	$s = 11^z_9$	$\log_9 s$	r
0	1	0	-
1	11	1.091329169	1.091329169
2	121	2.182658339	1.091329170
3	1331	3.273987508	1.091329169
4	14641	4.365316677	1.091329169
5	162151	5.459743807	1.094427130
6	1783661	6.551072976	1.091329169
7	20731371	7.667472316	1.116399340
8	228145181	8.759001215	1.091528899
9	2520607101	9.852322719	1.093321504
10	27726678111	10.943651889	1.091329170



$$r = (\log_9 x^z) - (\log_9 x^{z-1}), \text{ for } x = 11_9$$

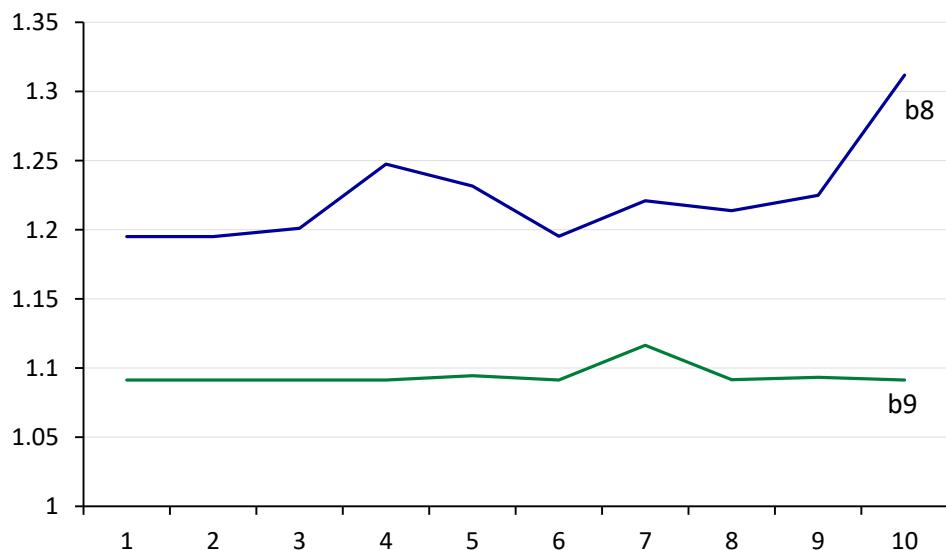
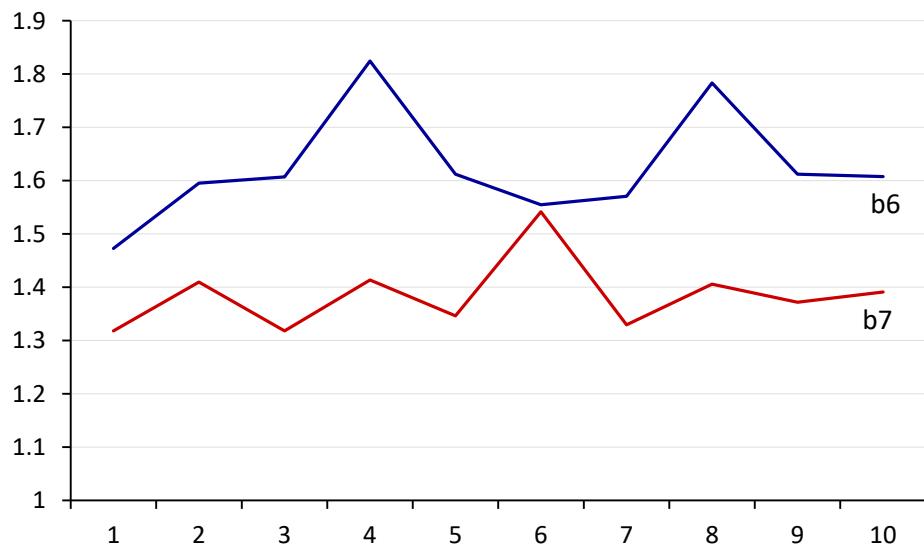
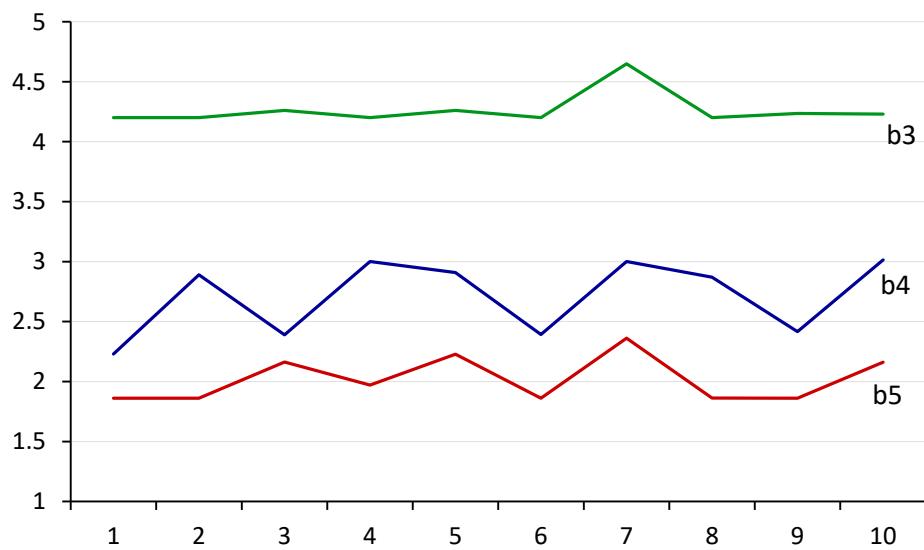
Proportional graphs

The first graph below shows the distributions represented on pages 5-12 above with a proportional vertical axis for the full range $b=(2, [...], 9)$. The three graphs on the subsequent page show the relationships between the distributions for the range $b=(3, [...], 9)$, with expanded vertical scales (r):



$$r = (\log_b x^z) - (\log_b x^{z-1}), \text{ for } x = (10_{10})_b$$

Graphs to show relations of close sequential groups



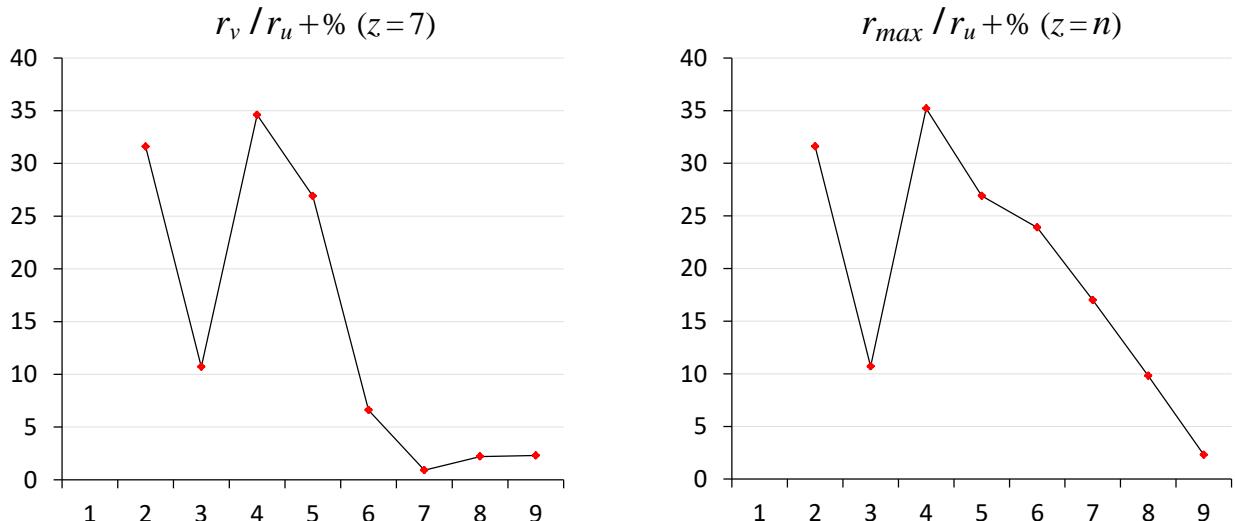
Variation factors

To measure the degrees of variation in proportion exhibited in the values of r in each radical series, I have taken as a baseline the value of r_u given for $z=1$, then calculated the increase factor for r_v at $z=7$ (as this appears as a frequent point of high elevation): (r_v/r_u) ; and, if $r_{max} > r_v$, the increase factor for r_{max} at $z=n$: (r_{max}/r_u) :

b	r_u	r_v	r_{max}	$r_v / r_u (z=7)$	$r_{max} / r_u (z=n)$
2	9.980139578	13.137426684	13.138464968	1.316	1.316 ($z=10$)
3	4.200863730	4.648524836	-	1.107	-
4	2.229715809	3.000611908	3.014650569	1.346	1.352 ($z=10$)
5	1.861353116	2.362122701	-	1.269	-
6	1.472885940	1.570507779	1.824276797	1.066	1.239 ($z=4$)
7	1.318123223	1.329620219	1.541601527	1.009	1.170 ($z=6$)
8	1.194987500	1.220911172	1.311822583	1.022	1.098 ($z=10$)
9	1.091329169	1.116399340	-	1.023	-

$$[r = (\log_b x^z) - (\log_b x^{z-1}), \text{ for } x=(10_{10})_b]$$

These variation factors are represented as percentage-increase as the vertical axis in the graphs below. The horizontal axis represents the value b . It is clear that the values for r_v / r_u and r_{max}/r_u are virtually identical for the radices represented by $b=(2, 3, 4, 5, \text{ and } 9)$, i.e., excepting $b=7$, and the two adjacent radices $b=6$ and $b=8$.



Analysis

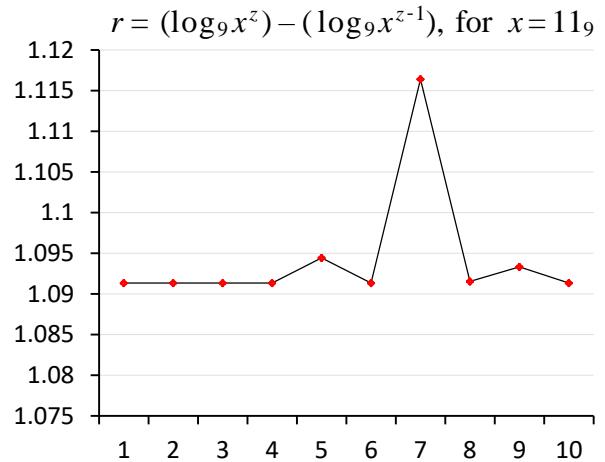
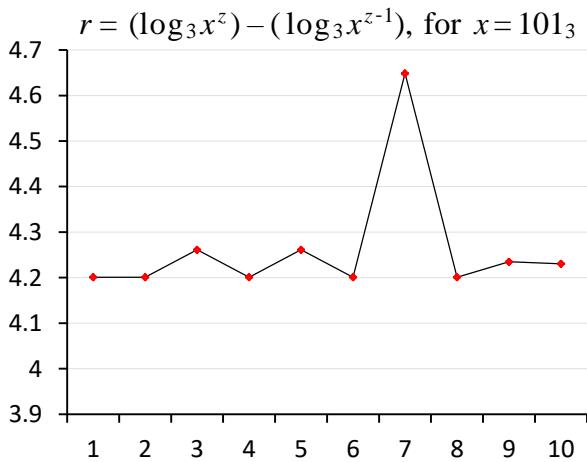
An earlier version of this document began from an initial comparison of values in the decimal exponential sequence: 10^z_{10} , for $z=(0, [...], 10)$, with their *octal* correspondents: 12^z_8 (re: p.11 above). It was then extended to similar comparisons for the series of numerical radices from binary to nonary, for values in each radix corresponding to 10 in decimal (re: pp.5-12). These exercises reveal that, while the ratios between successive values in a progressive exponential sequence (e.g. 10^z_{10} or 12^z_8) calculated according to the rules of the specific radix will be constant: $(x^n)_b / (x^{n-1})_b = (x)_b$, and would be represented graphically by a horizontal straight line at $y=(x)_b$ (i.e., with z as the horizontal axis); the treatment of the same ratios rendered through the logarithmic function (which derives all values from common logarithms – \log_{10}) results in series of ratios that are *no longer constant* – excepting of course those of base-10 itself. This is true for each of the numerical radices from binary to nonary, which are shown to be inconsistent with decimal, as well as being inconsistent with each other, and is evidenced by the graphs represented above, which in each case display a series of variegated peaks and troughs.

The same calculations performed for any value of x in decimal (without involving a derived radical logarithm), result in horizontal straight lines at $y=\log x$. In the exercises above there are no instances of horizontal straight lines; however, in later exercises, for values of x other than 10_{10} , horizontal straight lines in general result only where the decimal value of x is equal to the value b (see for example p.22 below, where $x=2, b=2$). There are three exceptions to this in the subsequent exercises: occurring where x is equal to b^2 or b^3 (see *Comments* on p.31 & p.51 below).

The logarithmic function was developed in the 17th century by John Napier (and later by Leonard Euler) on the premise of *invariant* proportion between integers, and only on this basis can the derivation of radical logarithms from ‘common’ logarithms be assumed to be unproblematic. For example, in the example of the octal series (p.11 above), the deviation from the horizontal affecting the values $z=3$ onwards is the result of applying logarithms to the octal series; i.e., the essential *proportionality* between integers on which the logarithmic function is based is lost. If we represent these ratios without the use of logarithms, i.e., by dividing successive exponentials in the series 12^z_8 along base-8 rules, we of course end up with a horizontal straight line at $y=12_8$. It is the derivation of the octal logarithmic values from ‘common’ decimal logarithms that introduces these deviations. This undermines the role of the logarithmic function in expressing *common ratios of proportion* across diverse number radices, and suggests that the rules of proportionality between integers apply only in a restricted sense, according to the particular range, or according to the ‘group characteristics’, of the select digits at our disposal. This failure inherent in the logarithmic function undermines the accepted principles of rational proportionality pertaining between diverse number radices and indicates that rationality operates effectively under formally circumscribed limits, where previously no such limits had been perceived. To account for this it is necessary to consider exactly what it is that defines an ‘integer’, as the behaviour of the

values in the octal series, as well as in other series, suggests that integers are unable to fulfil their customary role as stable indices of *intrinsic* value.

A comparison of the set of all eight series may help eliminate some erroneous or misleading explanations. The most striking comparison to make is in the shapes of the various distributions is that between the *ternary* (p.6) and *nonary* (p.12) series, as both series feature significant peaks at $z=7$, as well as featuring comparable smaller peaks to either side of $z=7$:



A comparison of the two elevations at $z=7$ in absolute terms shows the scale of the increase in the base-9 series is only $\approx 5.6\%$ of that in the base-3. However, in relative terms (i.e., taking into account the difference in the baseline value at $z=1$) the difference is between a $\approx 10.7\%$ increase in the ternary series, and a $\approx 2.3\%$ in the nonary, so that the latter increase is proportionally $\approx 21.6\%$ of the size of elevation in the ternary series.

This high individual peak is not a characteristic that is repeated in the other series however, with the exception of the *septenary* and *octal* series (p.10 & p.11) – in the former the peak occurring at $z=6$, with larger adjacent peaks than those in the previous examples. The *binary* (p.5), *quaternary* (p.7), *quinary* (p.8), and *senary* (p.9) series all feature more regular variations, with some signs of patterning, particularly in those of the *binary* and *quaternary* series. The *octal* series shows quite a unique distribution, especially in view of its elevated peak at $z=10$.

The statistical assessment of the variation factors on p.15 with respect to the maximum (r_{max}) and minimum (r_u) variations exhibited in each distribution, and their comparison with the variation factor at $z=7$ (r_v), shows that in four of the eight series ($b=2, 3, 5$, and 9) r_{max} is found to occur at r_v (for $b=2$, an equivalent maximum value recurs at $z=4, 7$, and 10 – the value of r at $z=10$ is only $\approx 0.008\%$ higher than that at $z=7$, which I have assumed to be negligible); while in the *quaternary* series ($b=4$) the increase factor at r_v is only $\approx 0.5\%$ lower than its maximum, which occurs at $z=10$. The effect of this is that the values for r_v/r_u , and those for r_{max}/r_u , are identical across 4/8 of the series, and almost identical across 5/8 – the exceptions being $b=7$ and its two adjacent radices $b=6$ and $b=8$. This similarity is displayed in the two graphs on p.15 showing percentage increase factors for r_v/r_u and r_{max}/r_u (the

horizontal axis represents the value for b in these two graphs). For instance, where $b=4$ the value of r_v is a $\approx 34.6\%$ increase on the value of r_u .

The identity of shape across the greater part of these two distributions (see graphs on p.15) gives empirical proof of the qualitative uniqueness of the integer '7' (i.e., within the context of this particular series starting from the decimal $x=10$)*, and which is irreducible to rational or quantitative principles. A further observation is that in 3/8 of the series ($b=2$, $b=4$, and $b=8$) r_{max} occurs at $z=10$ (n.b., in $b=2$ and $b=4$, r_{max} and r_v are virtually identical), so that both $z=7$ and $z=10$ appear to be 'potentiated' in comparison to other values of z . It is noticeable also that the distribution of peaks where $b=2$ (i.e., at $z=4$, $z=7$, and $z=10$) is also a characteristic found where $b=4$, and where $b=8$, although in the latter case the peaks do not exhibit the regularity of elevation found in the former two cases.

Viewing the set of distributions as a whole, it is noticeable that there is an absence of close correspondence or consistency between any two distributions (where they exhibit similarities of shape for instance, they disagree in scale). However, the statistical analysis reveals the frequency of $z=7$ as the point at which a half of the distributions reach maximum variation, and at which an additional one is very nearly at its maximum. The recurrent significance of $z=7$ and $z=10$ therefore seems to exclude any explanation of variation in terms of random or chaotic factors – an explanation which is also resisted by evidence of patterning in some of the distributions.

In the notations we have employed in this document there are two values – z and b – which represent series of progressive whole numbers. In the conventional understanding of the meaning of an 'integer' (i.e., an *entity* which is self-contained, qualitatively neutral, and whose value is determined intrinsically) we should expect such series to behave proportionally. If we limit our calculations to decimal notation, there might never be an indication that integers behave in any other way (this possibly explains how these phenomena have escaped the attention of mathematicians during the 400 years since the development of Napier's method). However, in the comparisons outlined above, there are exposed proportional inconsistencies both in the sequence $z=(0, [...], 10)$, and in the sequence $b=(2, [...], 9)$ – the former by comparing the logarithmic differences between sequential values in any individual radical sequence, and the latter in the absence of formal consistency between the shapes of the graphs of successive radical distributions (compare, for instance, the distributions for $b=6$ and $b=7$ on p.14 above).

* The same qualitative characteristics do not apply to instances of $z=7$ in the series taking the decimal values of x from 2 to 9 as their starting point, as shown in the analysis of variations across these further series below (pp. 102-112).

Numbers: Transcendental Objects or Symbolic Constructs?

I have discussed elsewhere some of the theoretical problems with the concept of integers (see: *Integers & Proportion*).^{*} Integers are conventionally understood to represent marks of stable intrinsic value, on the basis that any integer can be broken down, proportionally, into its constituent single units. The integrity of the unit '1' is fundamental to this understanding – it is taken as an objective unit of quantity, capable of expressing any larger quantity unambiguously by simple addition or multiplication. The problem with this understanding is that the unit '1' is not a phenomenal object with measurable properties *in itself*, but a symbolic construct, which 'stands-in' by a sort of tacit mental agreement as an index for value. As such it is a character that lacks a stable substantial basis, unless, that is, we assert that certain mental constructs possess transcendental objectivity. If we consider the unit '1' in the context of binary notation, for instance, we perceive that in addition to its quantitative value it is also invested with the *quality* of 'positivity', it being the only alternative character to the 'negative' '0'.[†] Can such qualitative properties be contained within the transcendental objectivity of the unit '1', considering that they do not similarly apply to '1' in decimal? In this case clearly not, as the property arises only as a condition of the restrictive binary relationship between the two digits. Hence it appears as a necessary conclusion that there are dynamic, context-specific attributes associated with particular integers which are not absolute or fixed (intrinsic), but variable, and which are determined extrinsically, according to the *relative frequency* of individual elements within the restrictive range of available characters circumscribed by the terms of the current working numerical radix.

The results that have been elaborated above in the comparisons of diverse number radices add empirical weight to this critique. Numbers, it appears, are subject to *behaviours* according to their relative position within a limited series of available digits; that is, according to their *relative values*. It makes a difference to an instance of '1', in terms of its relative frequency, or its *dispositional value*, whether it is 1 in base-3 or 1 in base-10, for instance, even though 1_3 and 1_{10} are quantitatively identical. I have tried to show also, in the statistical assessment of sequential radical distributions (re: pp.15-18 above), how this instability does not arise from random or chaotic factors, but rather determines certain patterns of affinity within a given context of variation (as exemplified by the heightened potential of $z=7$ within the context of $x=10$).[‡] This is the case, as I understand it, because numbers, rather than behaving as stable transcendental objects, are affected by context-specific dynamic and dispositional properties, as well as by rational principles. In fact, rational principles – the ability of numerical values to express conformable *ratios* – do not operate as universal principles consistently governing all

* Published at: <http://somr.info/xcetera/integer.php>.

† This characteristic is of course what enables digital computer systems to employ binary code principally to convey a series of processing instructions, rather than merely as an index of quantity.

‡ As a further example, see the distributions of variation factors given for the series where $x=9$, on p.110 below. In this context there is displayed a heightened frequency of elevated values of r where $z=6$, occurring in 4/8 of the series (i.e., for $b=2, 4, 5, \& 8$).

numerical notations, but depend locally on the terms of the particular numerical radix we happen to be employing. The rational principles governing a decimal system are incompatible with those governing an octal or a binary one, as is clearly shown in the distributions exhibited in the preceding pages.

Clearly, issues of relative frequency must be determined *extrinsically* – from the relations between integers as notional quantities, or between the individual members of relational groups of digits (as quantities) in series. From the conventional rational (and, it has to be said, deeply metaphysical) viewpoint of integers as self-contained transcendental objects whose values are determined intrinsically, the constitutive effect of numerical behaviour upon value, of quality upon quantity, will forever be shrouded in mystery.

These criticisms are not intended to be exhaustive, but are an initial response to the problems arising from the failure of the principle of rational proportionality, as evinced by the exercises in the foregoing, as well as in the subsequent, sections. I do not suggest here that this analysis provides a full or adequate explanation of the instances of variable proportionality revealed in these exercises. Attempts at analytical explanation tend to be predisposed towards rational solutions; and after all it was necessary to employ the distinctly rational method of logarithmic comparison in order to expose the failures in rational proportionality as the principle underpinning that method.

Having got thus far, and in the light of these results, it may be that henceforth we must accept that rationality operates only within well-defined limits – that it is ultimately undermined as an overarching principle in the analysis of quantitative systems. It begins to appear as a *contingent*, relational property, which fluctuates according the terms of a signifying regime (i.e., according to its numeric or indexical syntax), rather than as a *necessary* precondition with universal, or absolute, applicability. This suggests that a fully rational explanation of these results may ultimately be unjustifiable, or inappropriate.

Further historical and epistemological enquiry should help towards an understanding of how the selective inheritance of certain notions from Classical mathematics lent support to the overvaluation of the principle of rational proportionality by hypostatising that which should correctly be understood as a fundamentally conceptual category (number), into something idealised as a concrete entity (integer), asserting the latter as proportionally invariant, or qualitatively indifferent. A major impulse, from the 16th Century onwards, towards the mechanical understanding of the laws of Nature, required the establishment of systems of absolute measurement and quantification; which in turn enforced the rational dichotomy between quantitative and qualitative forms of knowledge. While this may have been historically and theoretically necessary to the development of modern empirical science in its infancy, from our own developed information-based technological perspective, which mistakenly assumes a seamless correspondence between diverse numerical radices (binary, octal, decimal, hexadecimal, base-64, etc.), the continued disregard for their rational non-conformability becomes an issue of urgent critical importance.

Extending the Series

The permutations of the series of radical distributions presented above have all taken the decimal value $x=10$ as their starting point, in order to make comparative examinations of the differences between successive exponential values of x in terms of r , that is, by converting the series $x=(10^0, [...], 10^{10})$ into its corresponding values for number radices between 2 and 9, as expressed by the value b .

In order to extend the comparisons for other values of x , but to restrict x to the notation of decimal-only integers, let's establish the value y , such that $y=(x)_b$, that is, x (a positive *decimal* whole number) converted to base- b .

According to this nomenclature, r is expressed as:

$$r = (\log_b y^z) - (\log_b y^{z-1}) \quad [\log_b y^z = \log_{10} y^z / \log_{10} b]$$

The following pages are work in progress, but initially they repeat the permutations already made in respect of $x=10$, for the successive values of x shown in the top row of the table below:

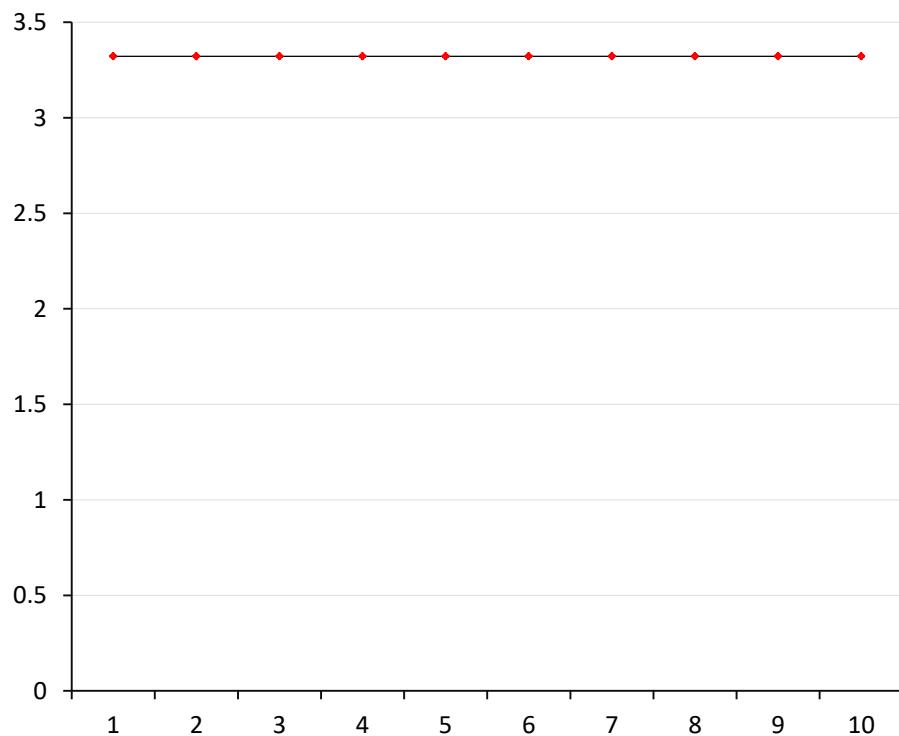
z	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$	$x=8$	$x=9$
0	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
2	4	9	16	25	36	49	64	81
3	8	27	64	125	216	343	512	729
4	16	81	256	625	1296	2401	4096	6561
5	32	243	1024	3125	7776	16807	32768	59049
6	64	729	4096	15625	46656	117649	262144	531441
7	128	2187	16384	78125	279936	823543	2097152	4782969
8	256	6561	65536	390625	1679616	5764801	16777216	43046721
9	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401

The exponential values shown in the above table serve as the basis for their radical conversions as y^z in the exercises that occupy pages 22-101 below. With respect to each successive value of $x=(2, [...], 9)$, the following subsections display the values for $\log_b y^z$ and r for the radical series $b=(2, [...], 9)$. Graphical representations of r against z are displayed as vertical and horizontal axes respectively (n.b. the vertical axes in these graphs are not at a constant scale).

$$\underline{\chi=2}$$

Binary

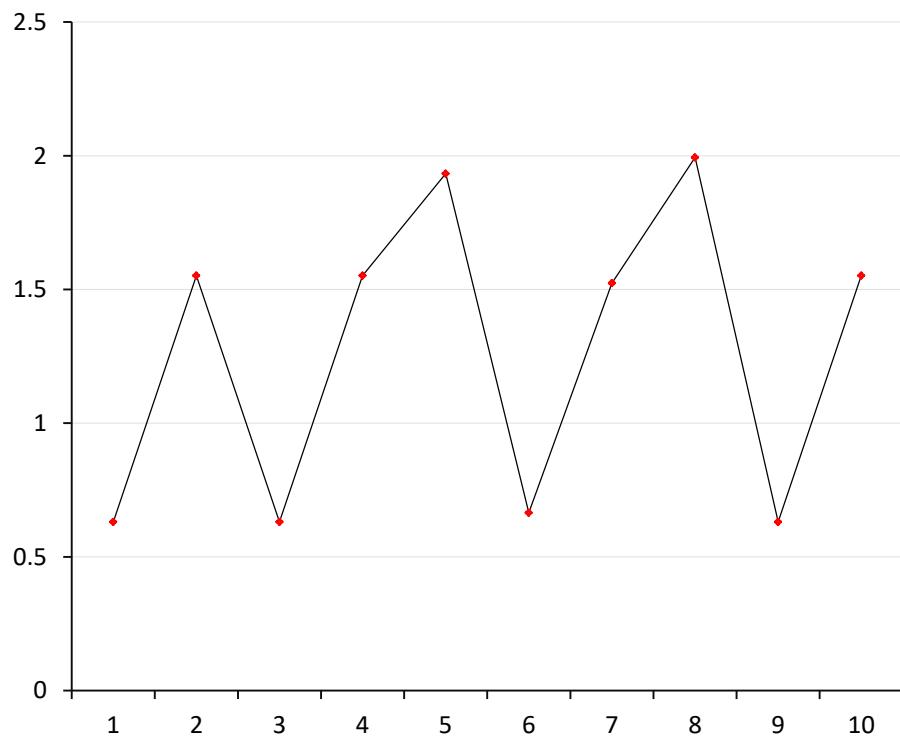
z	$y^z [x=2, b=2]$	$\log_2 y^z$	r
0	1	0	-
1	10	3.321928095	3.321928095
2	100	6.643856190	3.321928095
3	1000	9.965784285	3.321928095
4	10000	13.287712380	3.321928095
5	100000	16.609640474	3.321928094
6	1000000	19.931568569	3.321928095
7	10000000	23.253496664	3.321928095
8	100000000	26.575424759	3.321928095
9	1000000000	29.897352854	3.321928095
10	10000000000	33.219280949	3.321928095



$$r = (\log_2 y^z) - (\log_2 y^{z-1}), \text{ for } y = (2)_2$$

Ternary

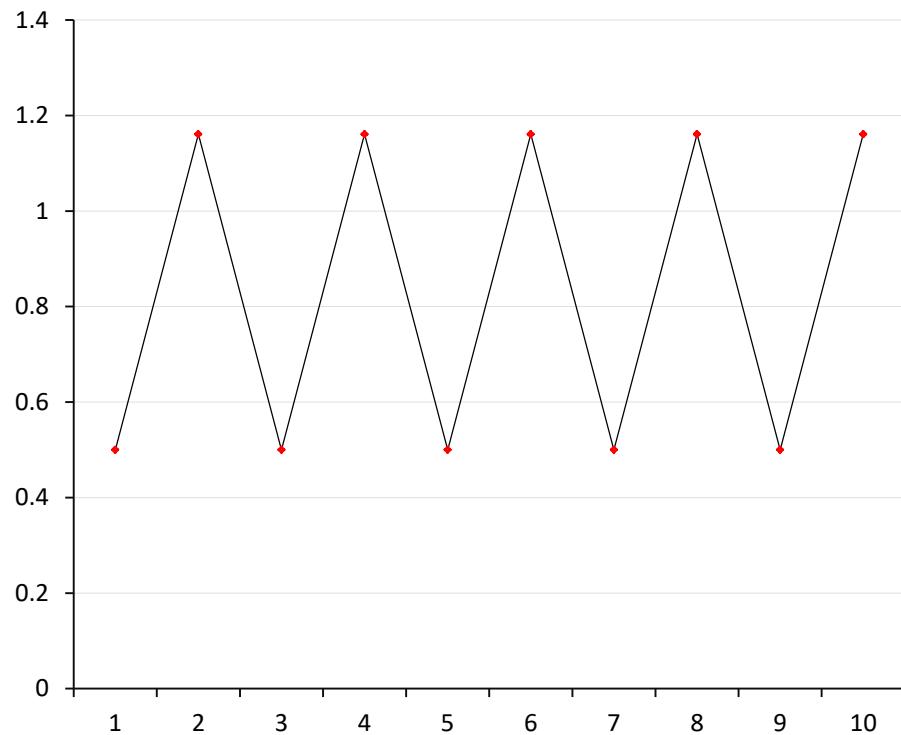
z	$y^z [x=2, b=3]$	$\log_3 y^z$	r
0	1	0	-
1	2	0.630929754	0.630929754
2	11	2.182658339	1.551728585
3	22	2.813588092	0.630929753
4	121	4.365316677	1.551728585
5	1012	6.298567676	1.933250999
6	2101	6.963483642	0.664915966
7	11202	8.486931840	1.523448198
8	100111	10.480526177	1.993594337
9	200222	11.111455930	0.630929753
10	1101221	12.663184515	1.551728585



$$r = (\log_3 y^z) - (\log_3 y^{z-1}), \text{ for } y = (2)_3$$

Quaternary

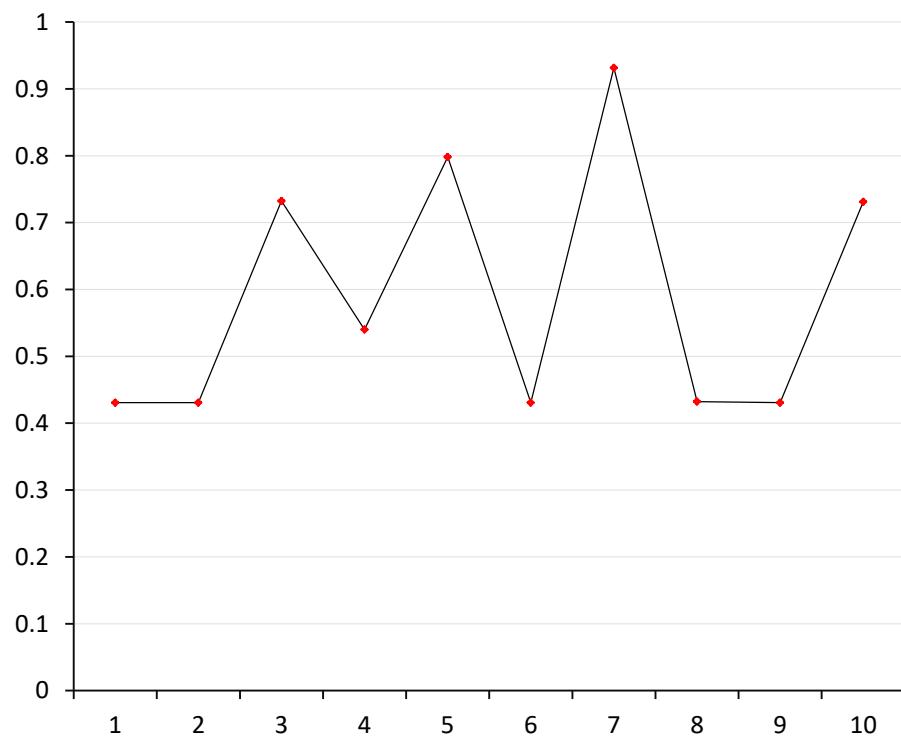
z	$y^z [x=2, b=4]$	$\log_4 y^z$	r
0	1	0	-
1	2	0.5	0.5
2	10	1.660964047	1.160964047
3	20	2.160964047	0.5
4	100	3.321928095	1.160964048
5	200	3.821928095	0.5
6	1000	4.982892142	1.160964047
7	2000	5.482892142	0.5
8	10000	6.643856190	1.160964048
9	20000	7.143856190	0.5
10	100000	8.304820237	1.160964047



$$r = (\log_4 y^z) - (\log_4 y^{z-1}), \text{ for } y = (2)_4$$

Quinary

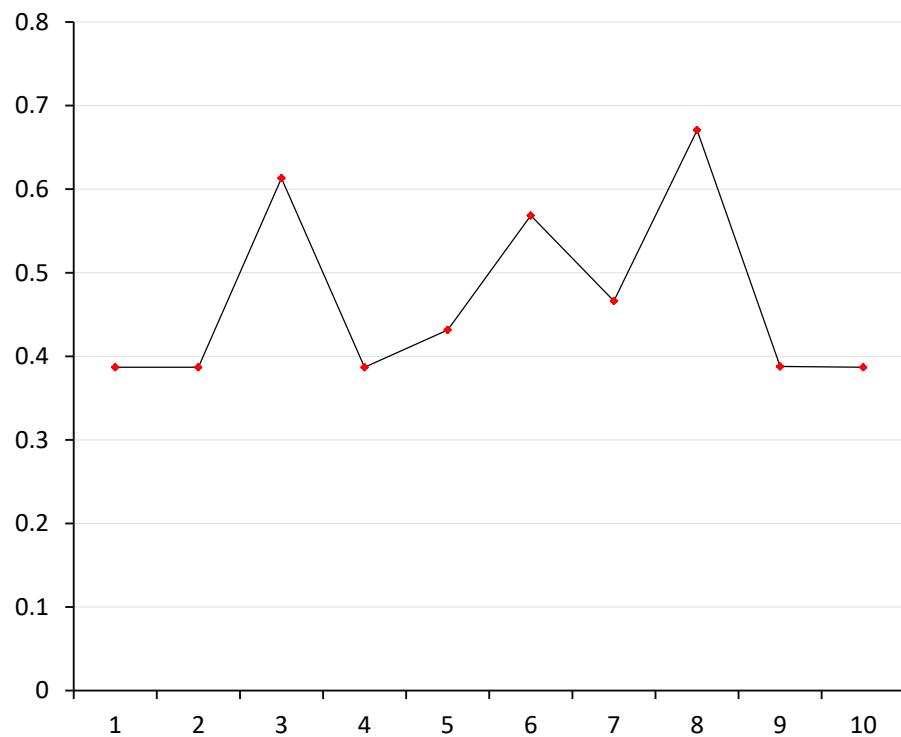
z	$y^z [x=2, b=5]$	$\log_5 y^z$	r
0	1	0	-
1	2	0.430676558	0.430676558
2	4	0.861353116	0.430676558
3	13	1.593692641	0.732339525
4	31	2.133656215	0.539963574
5	112	2.931768187	0.798111972
6	224	3.362444745	0.430676558
7	1003	4.293890889	0.931446144
8	2011	4.726114211	0.432223322
9	4022	5.156790769	0.430676558
10	13044	5.887821744	0.731030975



$$r = (\log_5 y^z) - (\log_5 y^{z-1}), \text{ for } y = (2)_5$$

Senary

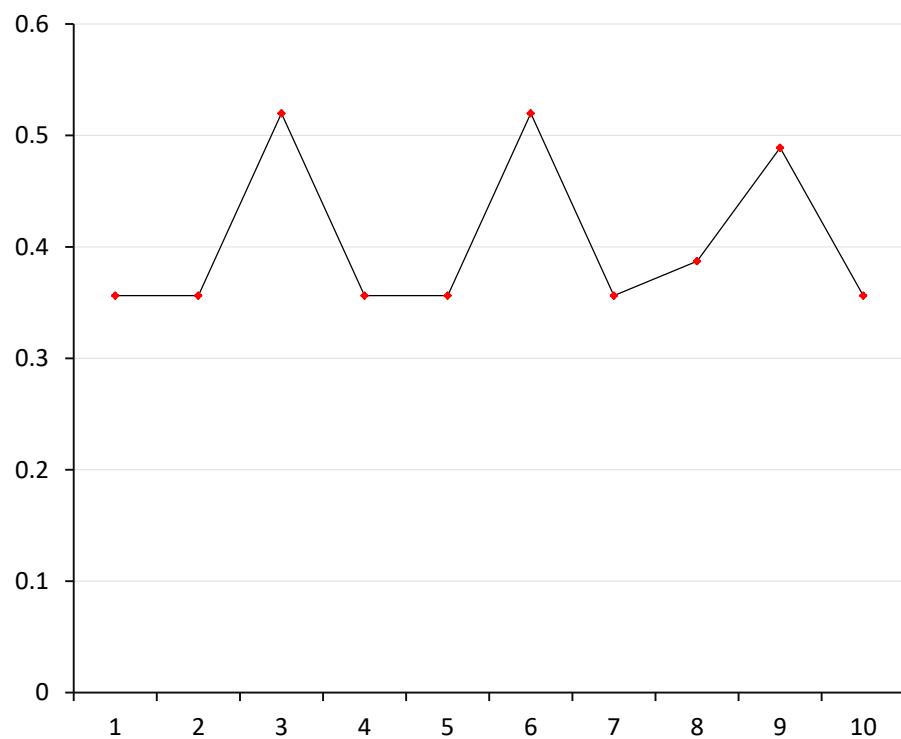
z	$y^z [x=2, b=6]$	$\log_6 y^z$	r
0	1	0	-
1	2	0.386852807	0.386852807
2	4	0.773705614	0.386852807
3	12	1.386852807	0.613147193
4	24	1.773705614	0.386852807
5	52	2.205231107	0.431525493
6	144	2.773705614	0.568474507
7	332	3.239907515	0.466201901
8	1104	3.910511063	0.670603548
9	2212	4.298374026	0.387862963
10	4424	4.685226833	0.386852807



$$r = (\log_6 y^z) - (\log_6 y^{z-1}), \text{ for } y = (2)_6$$

Septenary

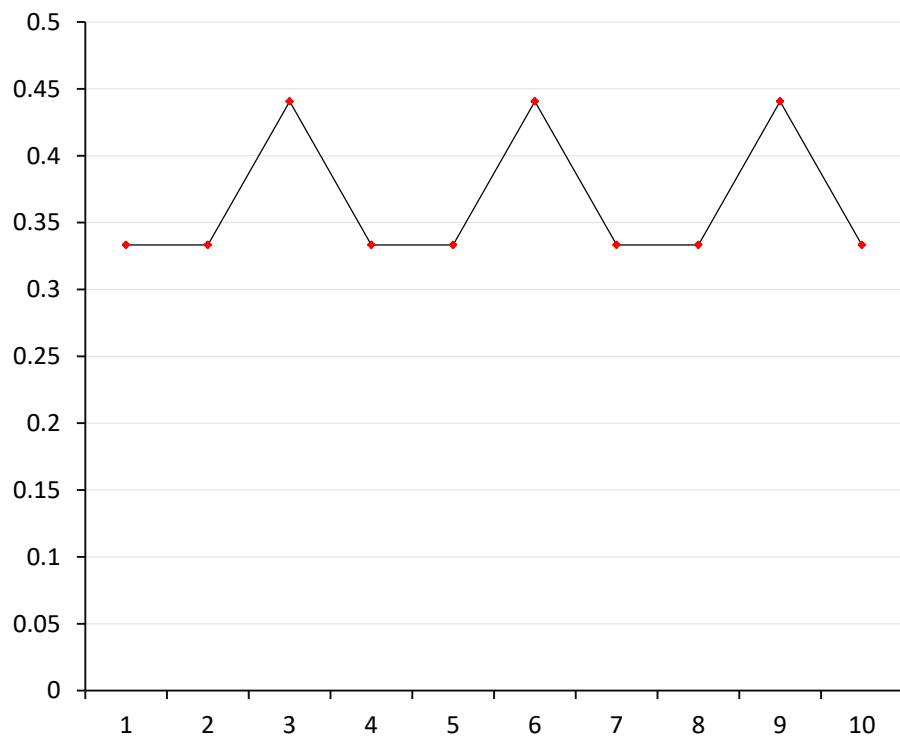
z	$y^z [x=2, b=7]$	$\log_7 y^z$	r
0	1	0	-
1	2	0.356207187	0.356207187
2	4	0.712414374	0.356207187
3	11	1.232274406	0.519860032
4	22	1.588481593	0.356207187
5	44	1.944688780	0.356207187
6	121	2.464548812	0.519860032
7	242	2.820755999	0.356207187
8	514	3.207868189	0.387112190
9	1331	3.696823218	0.488955029
10	2662	4.053030405	0.356207187



$$r = (\log_7 y^z) - (\log_7 y^{z-1}), \text{ for } y = (2)_7$$

Octal

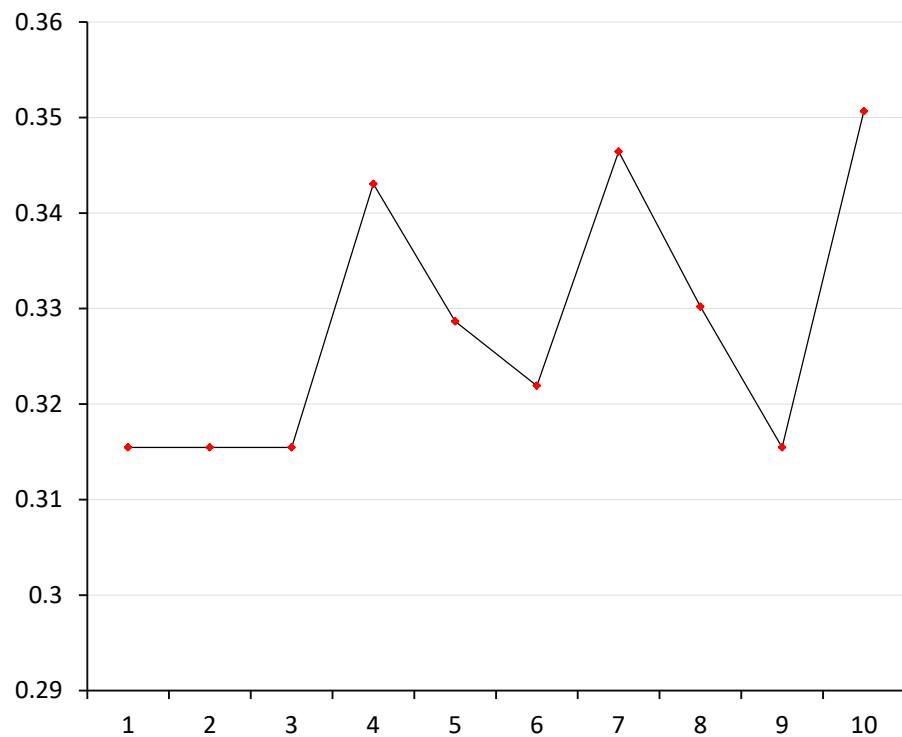
z	$y^z [x=2, b=8]$	$\log_8 y^z$	r
0	1	0	-
1	2	0.333333333	0.333333333
2	4	0.666666667	0.333333334
3	10	1.107309365	0.440642698
4	20	1.440642698	0.333333333
5	40	1.773976032	0.333333334
6	100	2.214618730	0.440642698
7	200	2.547952063	0.333333333
8	400	2.881285397	0.333333334
9	1000	3.321928095	0.440642698
10	2000	3.655261428	0.333333333



$$r = (\log_8 y^z) - (\log_8 y^{z-1}), \text{ for } y = (2)_8$$

Nonary

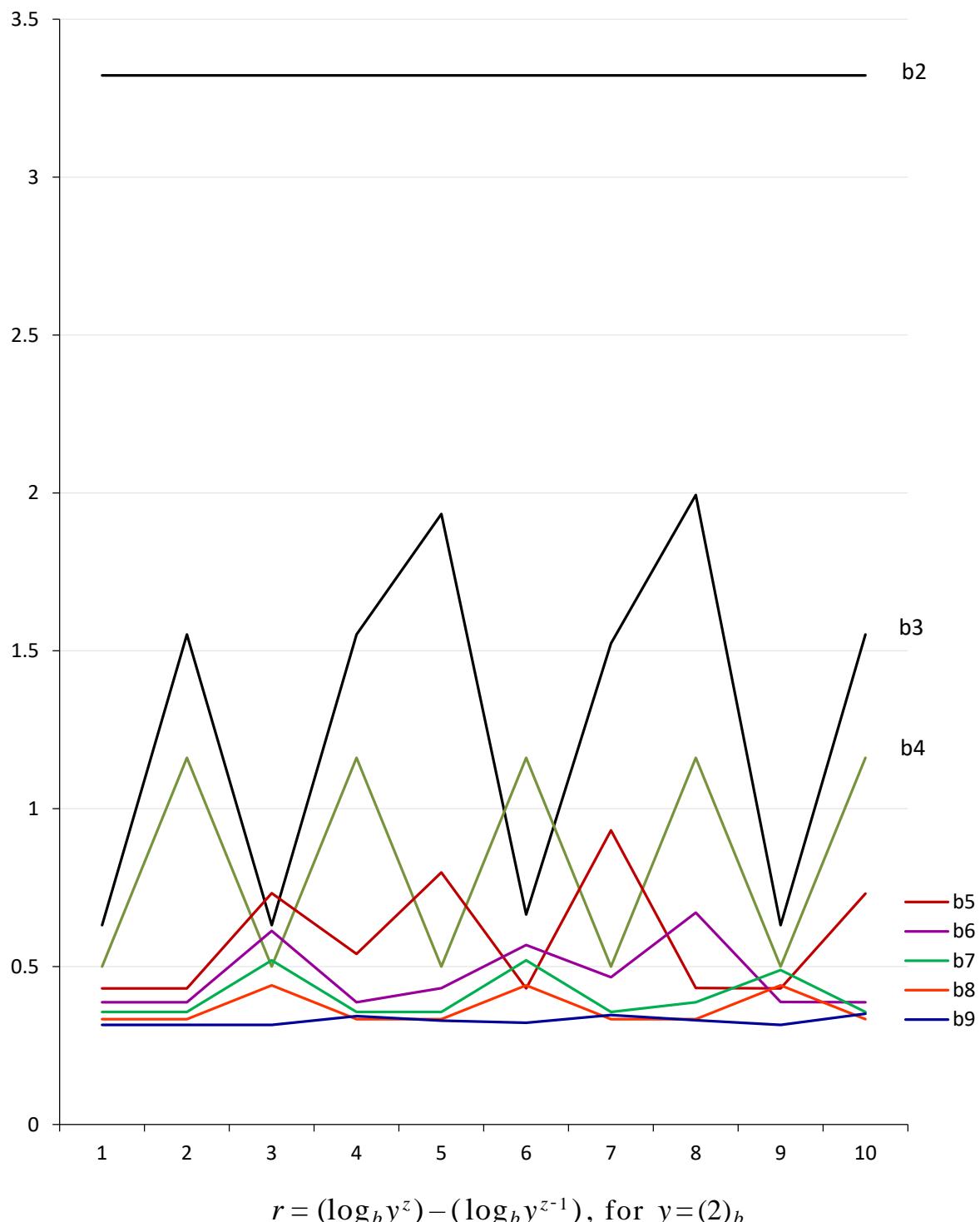
z	$y^z [x=2, b=9]$	$\log_9 y^z$	r
0	1	0	-
1	2	0.315464877	0.315464877
2	4	0.630929754	0.315464877
3	8	0.946394630	0.315464876
4	17	1.289450962	0.343056332
5	35	1.618108635	0.328657673
6	71	1.940029217	0.321920582
7	152	2.286466560	0.346437343
8	314	2.616661513	0.330194953
9	628	2.932126389	0.315464876
10	1357	3.282792180	0.350665791

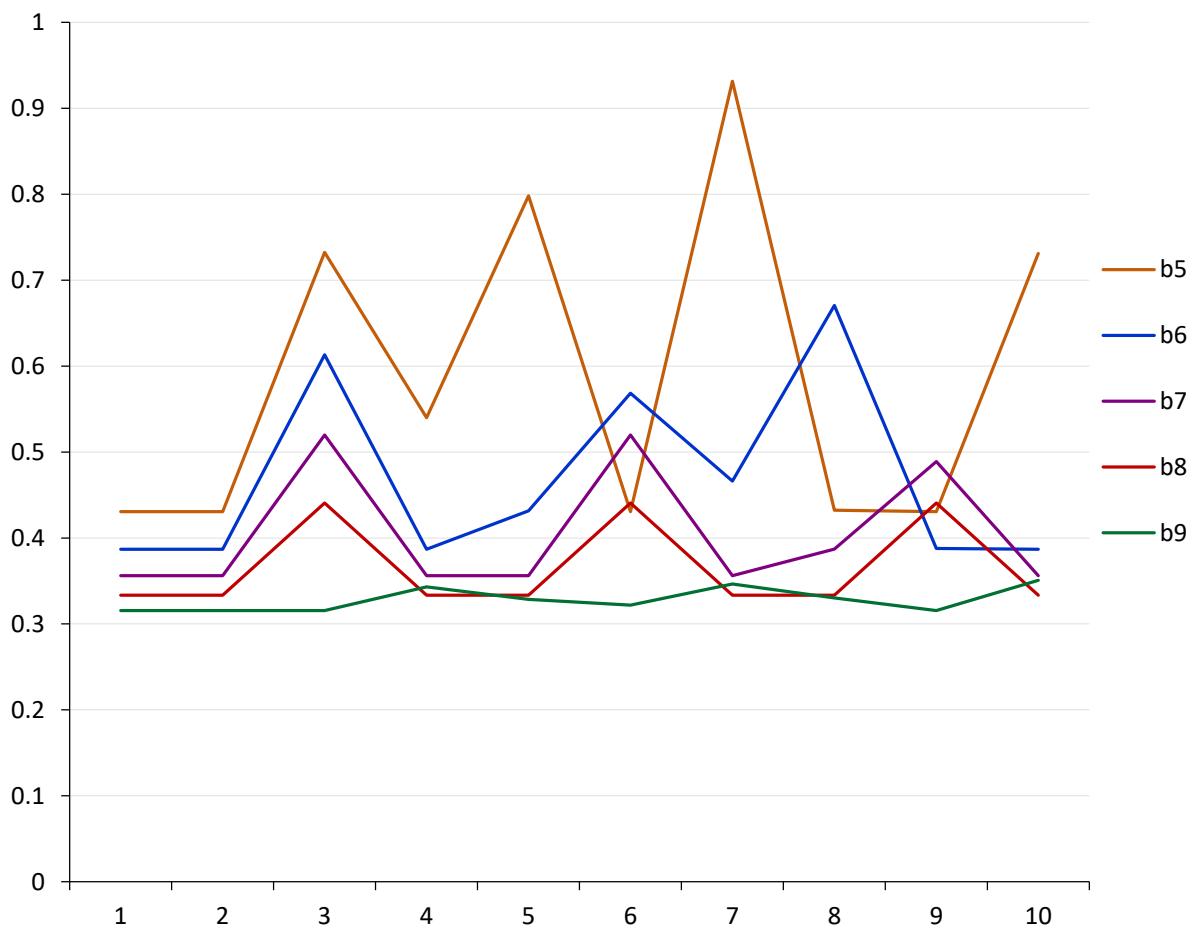


$$r = (\log_9 y^z) - (\log_9 y^{z-1}), \text{ for } y = (2)_9$$

Proportional graphs

The first graph below shows the distributions represented on pages 22-29 above with a proportional vertical axis for the full range $b=(2, \dots, 9)$. The second graph shows the relationships between the lower distributions for the range $b=(5, \dots, 9)$, with an expanded vertical scale (r):





Comments

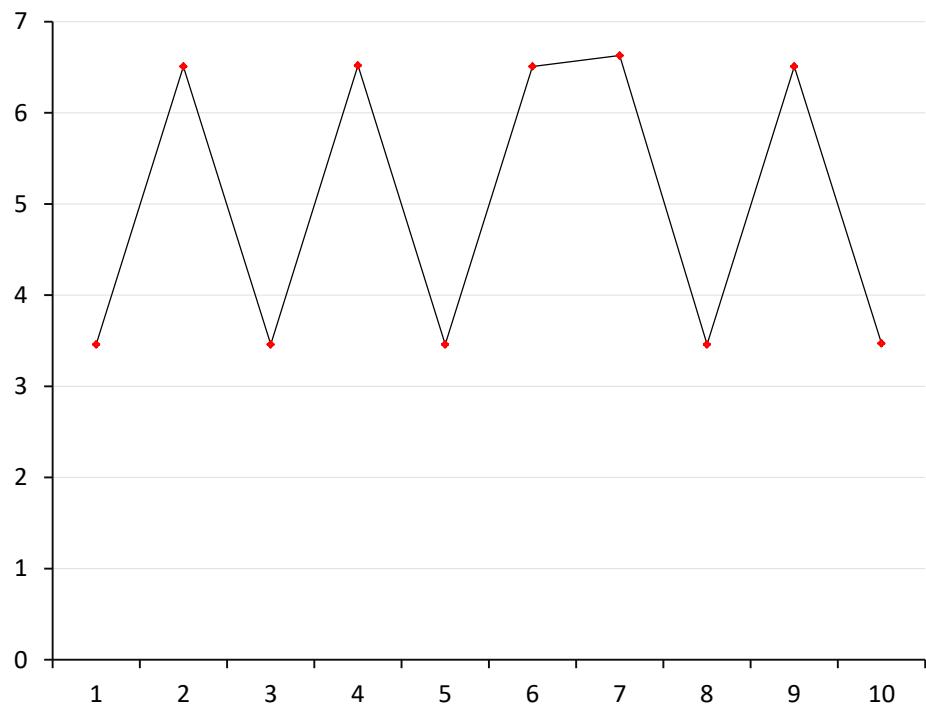
The first point to notice in examination of the results in the graph on p.30 is that for the *binary* series the values of r are consistent across the exponential series (≈ 3.32). Had we included calculations for base-10 in the range of values for b , we would see a similar horizontal display at $r = \log_{10}x$; i.e., for *all* values of x . It will become clear that a similar consistency applies to subsequent series where the *decimal* value x is equal to the value b (there are three exceptions to this in our results: two in the *binary* series (in the cases of $x=4$ and $x=8$ – see p.42 & p.82 below, and *Comments* on p.51), and a further in the *ternary* series (for $x=9$ – see p.93 below). In each unexceptional case (i.e., where $x=b$), $r = (\log_{10}x)^{-1}$. We might assume on this basis, as a general principle, that horizontal straight lines will occur where the decimal value of x (prior to its conversion to base- b) is equal to b , or to b^2 or b^3 (also, presumably, by extension to b^n).

The second point of interest is that of the extremity of the variations in the distributions between $b=3$ and $b=9$, as shown in the frequent overlapping and intersection of lines. For instance, where $b=5$, the elevation at $z=7$ represents an increase of 116% over against the initial value at $z=1$. Where $b=3$, the variation at $z=8$ is a 216% increase. Noticeable also is the conformity of some of the points of variation, e.g., at $z=3$, $z=6$, and $z=9$, in contrast to the reversal of the direction of variation at others (e.g., $z=6$, $b=5$).

$$\underline{x=3}$$

Binary

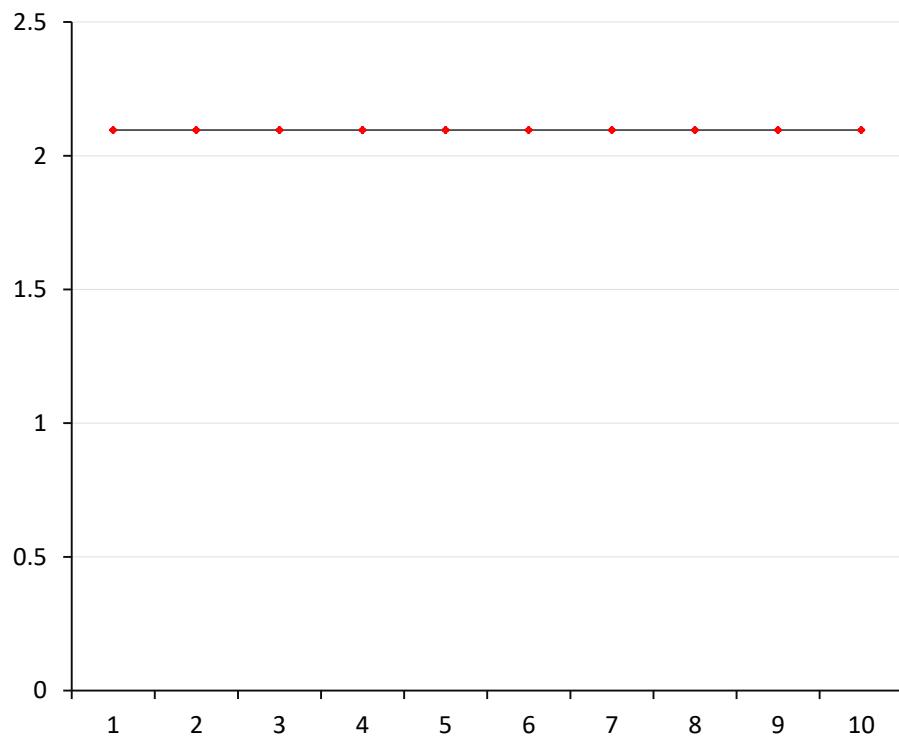
z	$y^z [x=3, b=2]$	$\log_2 y^z$	r
0	1	0	-
1	11	3.459431619	3.459431619
2	1001	9.967226259	6.507794640
3	11011	13.426657877	3.459431618
4	1010001	19.945925291	6.519267414
5	11110011	23.405356909	3.459431618
6	1011011001	29.913151550	6.507794641
7	100010001011	36.541353321	6.628201771
8	1100110100001	40.000785056	3.459431735
9	100110011100011	46.508579581	6.507794525
10	1110011010101001	49.979495410	3.470915829



$$r = (\log_2 y^z) - (\log_2 y^{z-1}), \text{ for } y = (3)_2$$

Ternary

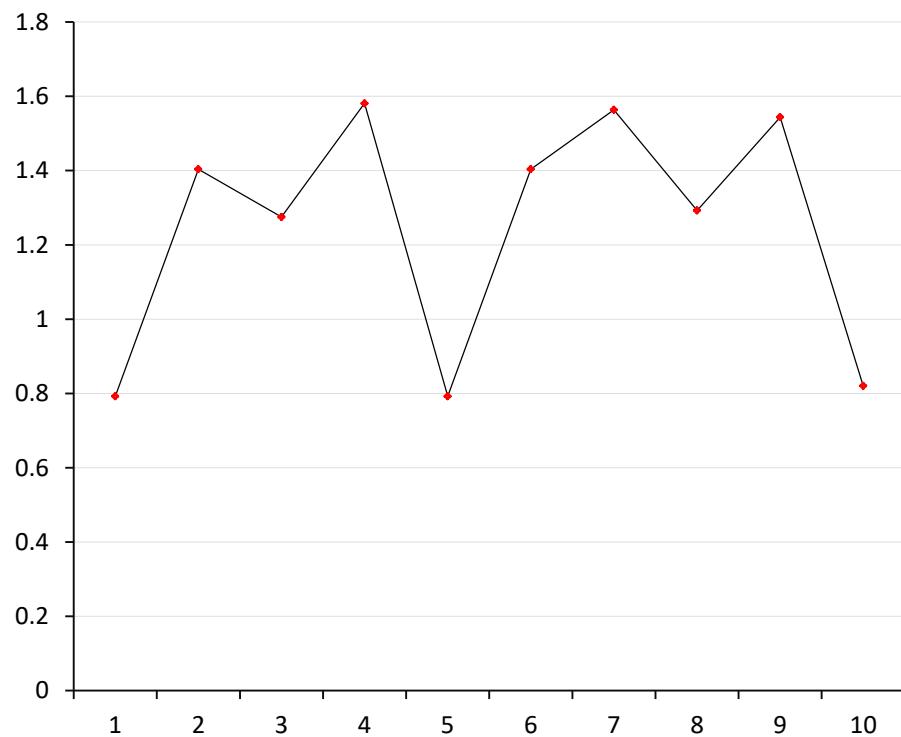
z	$y^z [x=3, b=3]$	$\log_3 y^z$	r
0	1	0	-
1	10	2.095903274	2.095903274
2	100	4.191806549	2.095903275
3	1000	6.287709823	2.095903274
4	10000	8.383613097	2.095903274
5	100000	10.479516371	2.095903274
6	1000000	12.575419646	2.095903275
7	10000000	14.671322920	2.095903274
8	100000000	16.767226194	2.095903274
9	1000000000	18.863129469	2.095903275
10	10000000000	20.959032743	2.095903274



$$r = (\log_3 y^z) - (\log_3 y^{z-1}), \text{ for } y = (3)_3$$

Quaternary

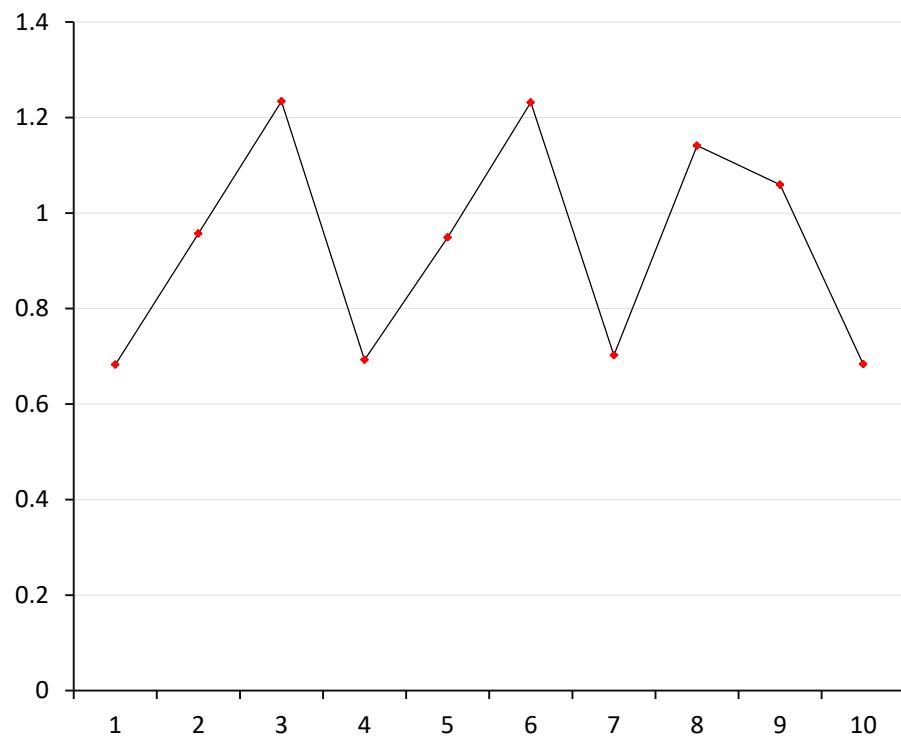
z	$y^z [x=3, b=4]$	$\log_4 y^z$	r
0	1	0	-
1	3	0.792481250	0.792481250
2	21	2.196158711	1.403677461
3	123	3.471257253	1.275098542
4	1101	5.052299377	1.581042124
5	3303	5.844780627	0.792481250
6	23121	7.248458088	1.403677461
7	202023	8.812080013	1.563621925
8	1212201	10.104598754	1.292518741
9	10303203	11.648294784	1.543696030
10	32122221	12.468534156	0.820239372



$$r = (\log_4 y^z) - (\log_4 y^{z-1}), \text{ for } y = (3)_4$$

Quinary

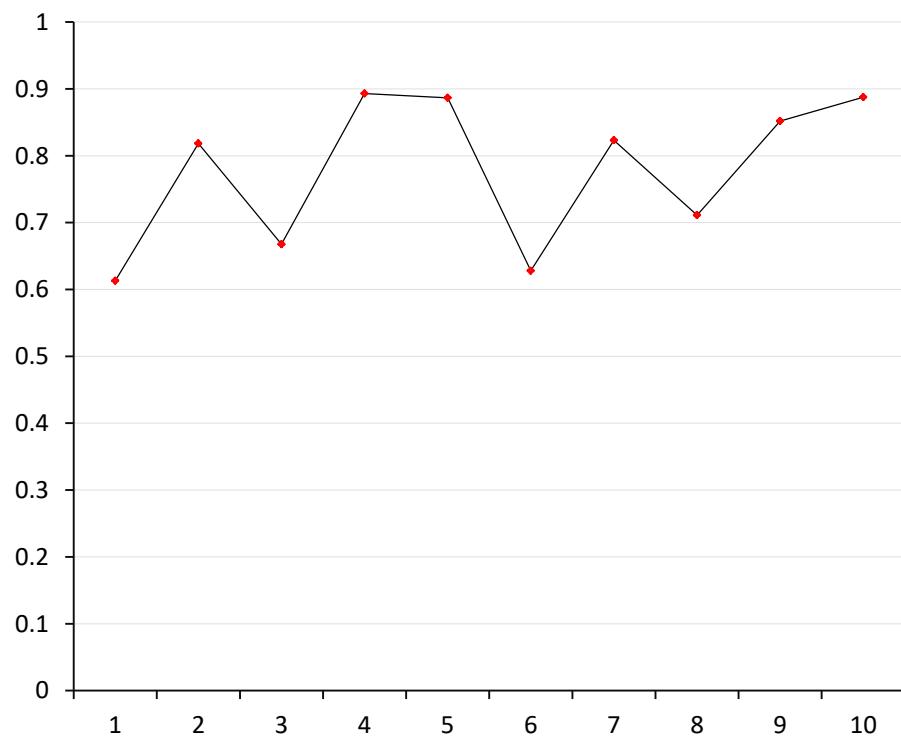
z	$y^z [x=3, b=5]$	$\log_5 y^z$	r
0	1	0	-
1	3	0.682606194	0.682606194
2	14	1.639738513	0.957132319
3	102	2.873657180	1.233918667
4	311	3.566333853	0.692676673
5	1433	4.515567436	0.949233583
6	10404	5.747314361	1.231746925
7	32222	6.449708092	0.702393731
8	202221	7.590921242	1.141213150
9	1112213	8.650139390	1.059218148
10	3342144	9.333769858	0.683630468



$$r = (\log_5 y^z) - (\log_5 y^{z-1}), \text{ for } y = (3)_5$$

Senary

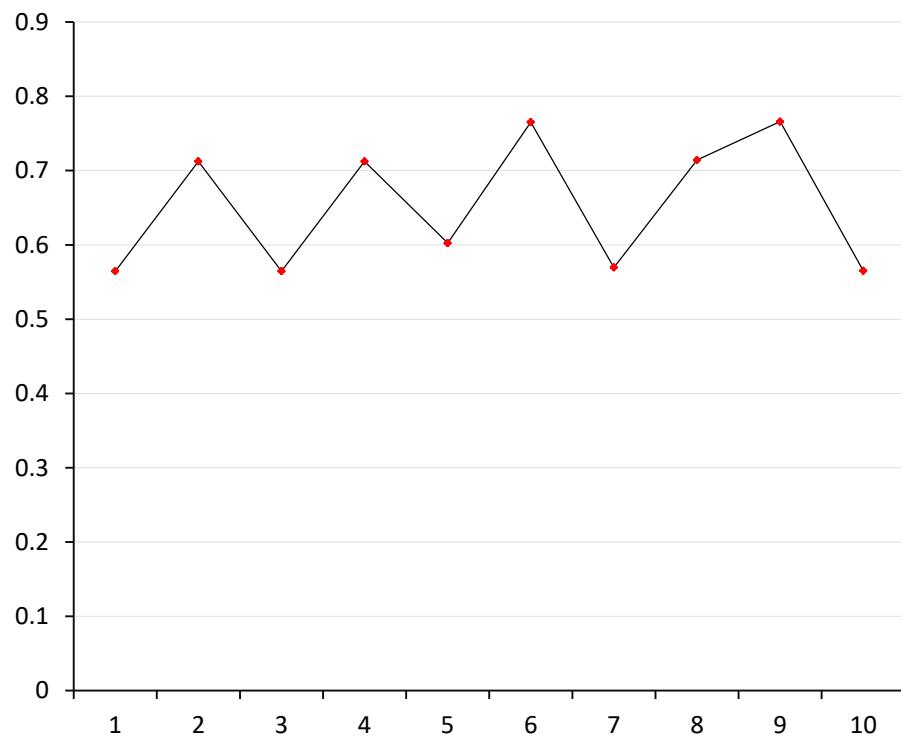
z	$y^z [x=3, b=6]$	$\log_6 y^z$	r
0	1	0	-
1	3	0.613147193	0.613147193
2	13	1.431525493	0.818378300
3	43	2.099165753	0.667640260
4	213	2.992194130	0.893028377
5	1043	3.878788741	0.886594611
6	3213	4.506721185	0.627932444
7	14043	5.329889136	0.823167951
8	50213	6.041005739	0.711116603
9	231043	6.892866666	0.851860927
10	1133213	7.780378872	0.887512206



$$r = (\log_6 y^z) - (\log_6 y^{z-1}), \text{ for } y = (3)_6$$

Septenary

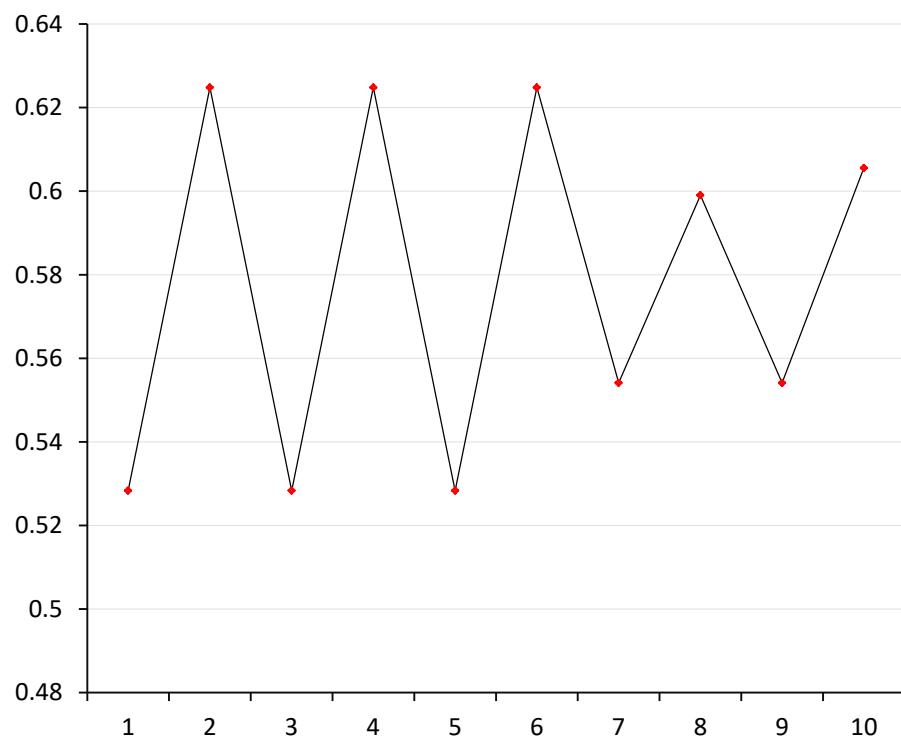
z	$y^z [x=3, b=7]$	$\log_7 y^z$	r
0	1	0	-
1	3	0.564575034	0.564575034
2	12	1.276989408	0.712414374
3	36	1.841564442	0.564575034
4	144	2.553978817	0.712414375
5	465	3.156382842	0.602404025
6	2061	3.921530799	0.765147957
7	6243	4.491068675	0.569537876
8	25062	5.205331828	0.714263153
9	111246	5.971241401	0.765909573
10	334104	6.536379702	0.565138301



$$r = (\log_7 y^z) - (\log_7 y^{z-1}), \text{ for } y = (3)_7$$

Octal

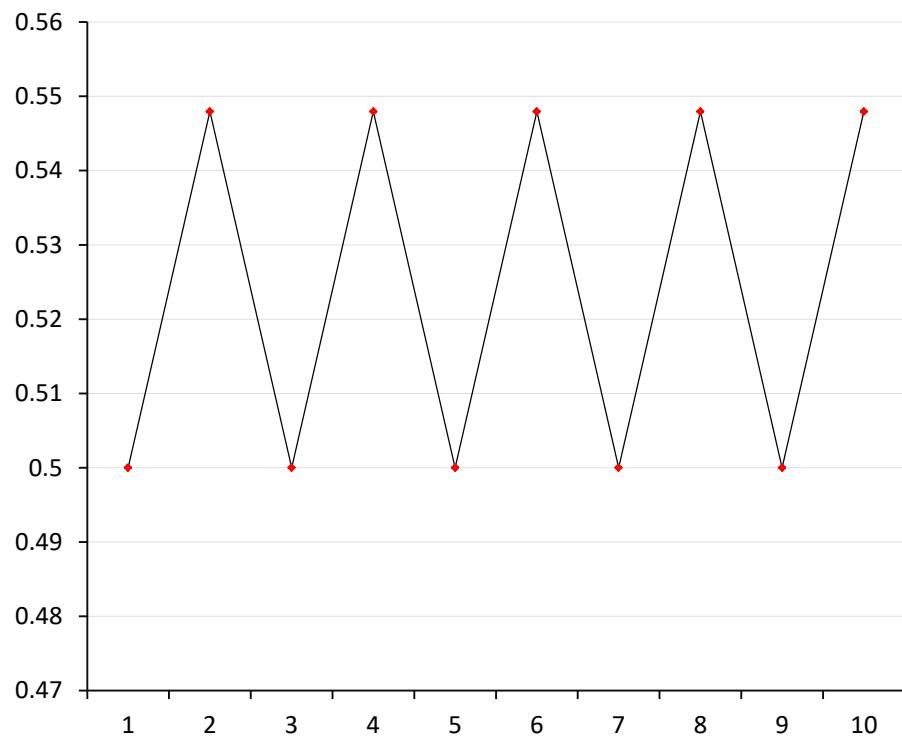
z	$y^z [x=3, b=8]$	$\log_8 y^z$	r
0	1	0	-
1	3	0.528320834	0.528320834
2	11	1.153143873	0.624823039
3	33	1.681464706	0.528320833
4	121	2.306287746	0.624823040
5	363	2.834608579	0.528320833
6	1331	3.459431619	0.624823040
7	4213	4.013544067	0.554112448
8	14641	4.612575492	0.599031425
9	46343	5.166687940	0.554112448
10	163251	5.772244101	0.605556161



$$r = (\log_8 y^z) - (\log_8 y^{z-1}), \text{ for } y = (3)_8$$

Nonary

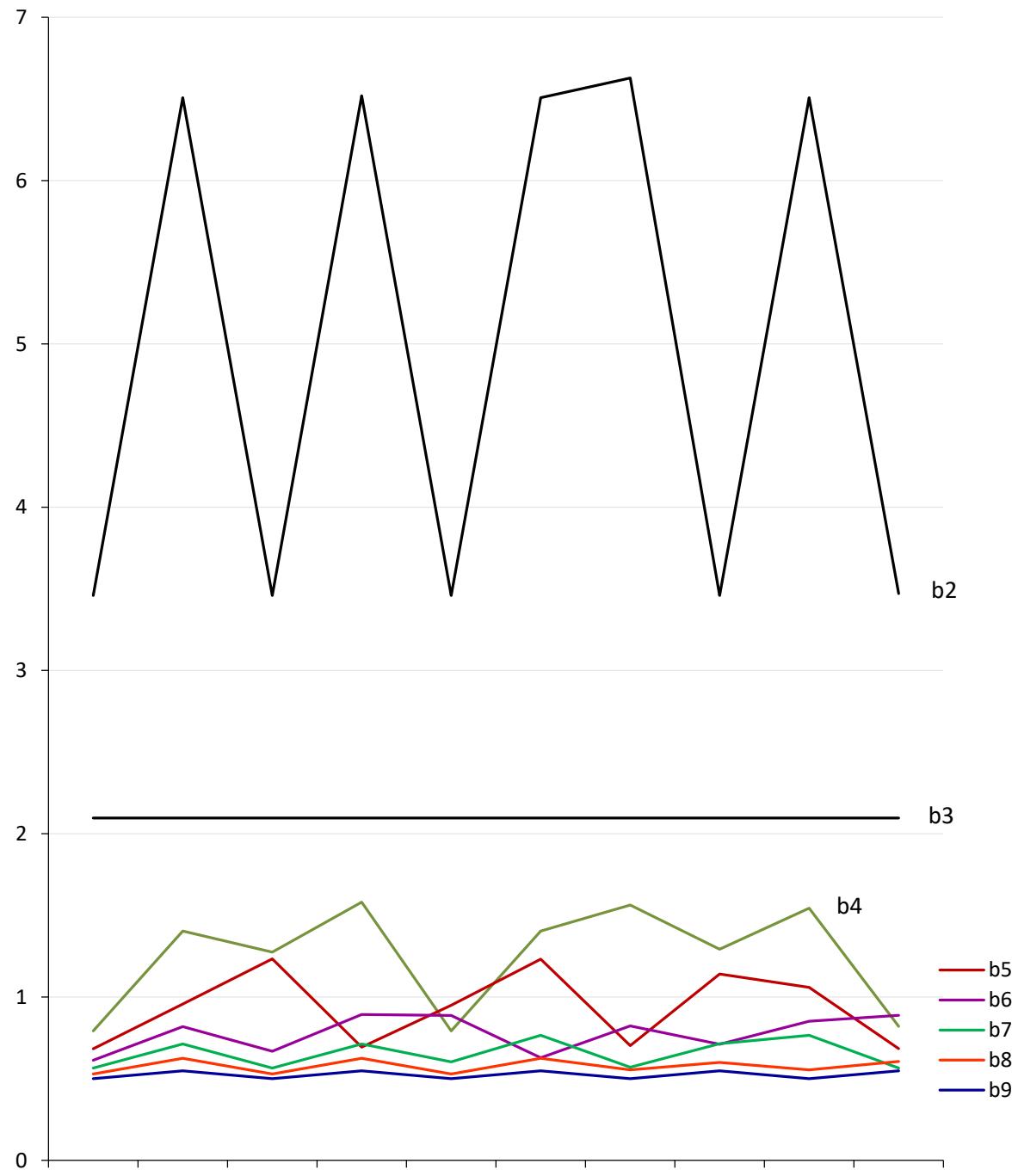
z	$y^z [x=3, b=9]$	$\log_9 y^z$	r
0	1	0	-
1	3	0.5	0.5
2	10	1.047951637	0.547951637
3	30	1.547951637	0.5
4	100	2.095903274	0.547951637
5	300	2.595903274	0.5
6	1000	3.143854911	0.547951637
7	3000	3.643854911	0.5
8	10000	4.191806549	0.547951638
9	30000	4.691806549	0.5
10	100000	5.239758186	0.547951637



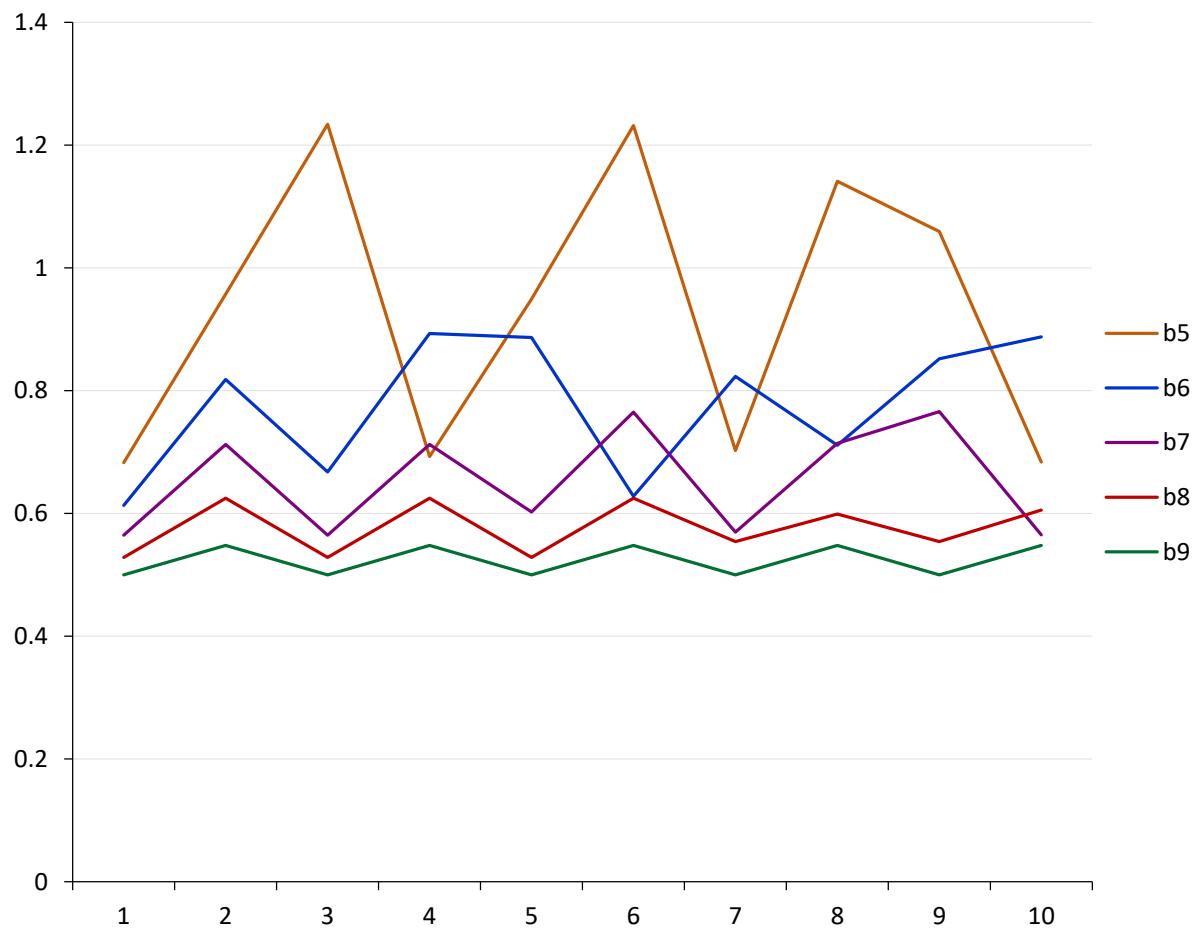
$$r = (\log_9 y^z) - (\log_9 y^{z-1}), \text{ for } y = (3)_9$$

Proportional graphs

The first graph below shows the distributions represented on pages 33-40 above with a proportional vertical axis for the full range $b=(2, \dots, 9)$. The second graph shows the relationships between the lower distributions for the range $b = (5, \dots, 9)$, with an expanded vertical scale (r):



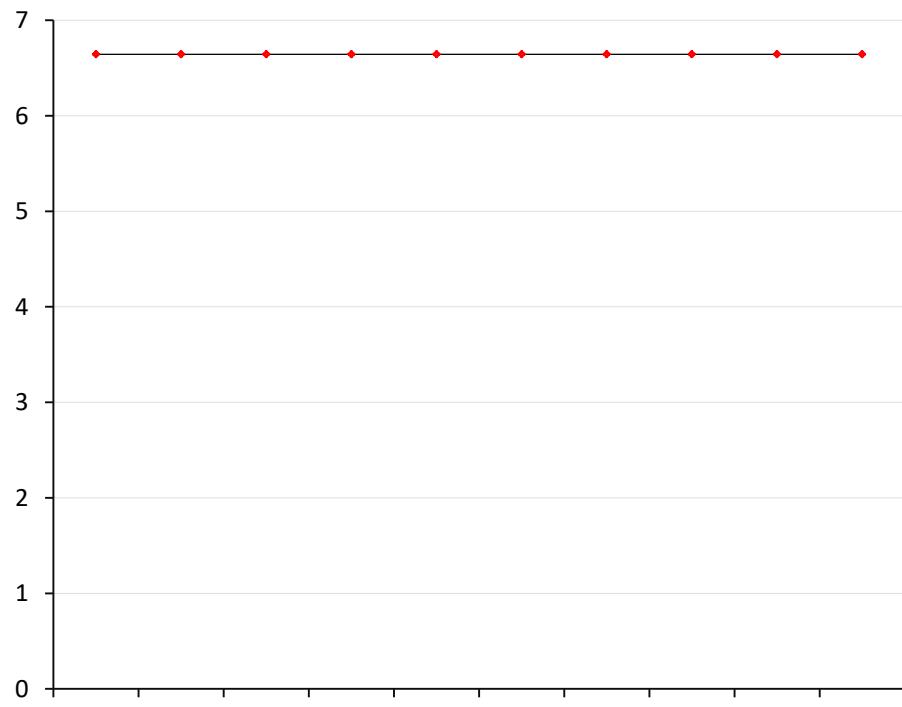
$$r = (\log_b y^z) - (\log_b y^{z-1}), \text{ for } y=(3)_b$$



$$\underline{x=4}$$

Binary

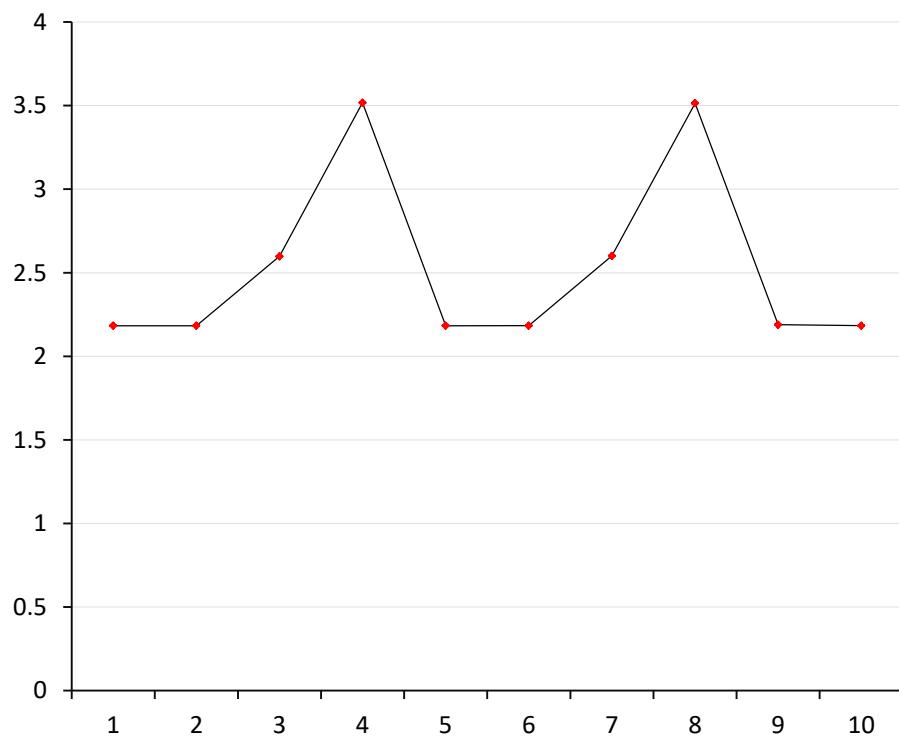
z	$y^z [x=4, b=2]$	$\log_2 y^z$	r
0	1	0	-
1	100	6.643856190	6.643856190
2	10000	13.287712380	6.643856190
3	1000000	19.931568569	6.643856189
4	100000000	26.575424759	6.643856190
5	100000000000	33.219280949	6.643856190
6	10000000000000	39.863137139	6.643856190
7	1000000000000000	46.506993328	6.643856189
8	10000000000000000	53.150849518	6.643856190
9	100000000000000000	59.794705708	6.643856190
10	1000000000000000000	66.438561898	6.643856190



$$r = (\log_2 y^z) - (\log_2 y^{z-1}), \text{ for } y = (4)_2$$

Ternary

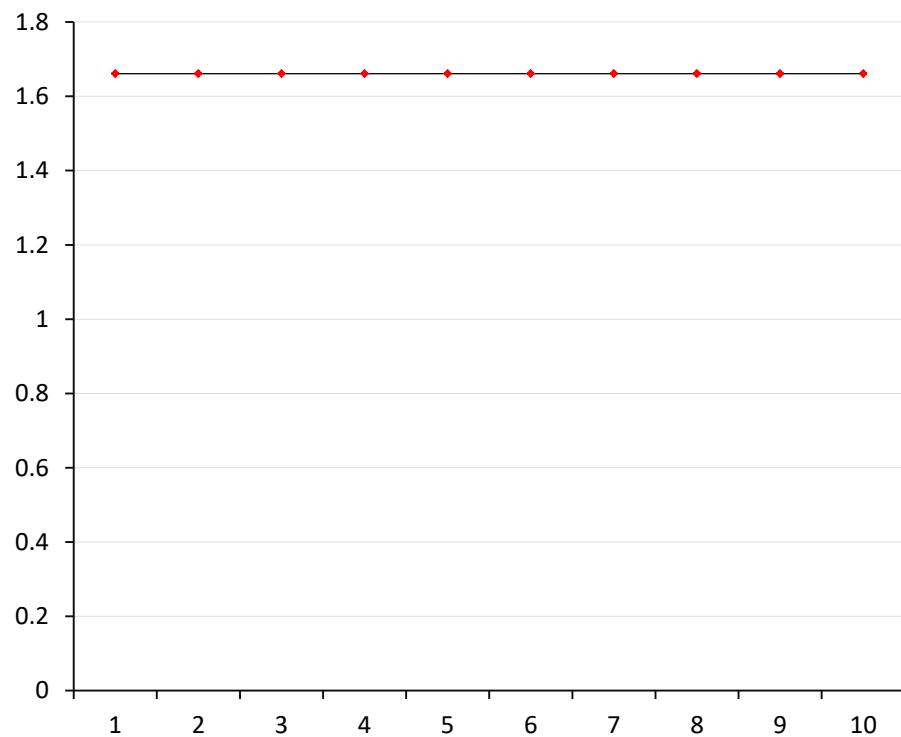
z	$y^z [x=4, b=3]$	$\log_3 y^z$	r
0	1	0	-
1	11	2.182658339	2.182658339
2	121	4.365316677	2.182658338
3	2101	6.963483642	2.598166965
4	100111	10.480526177	3.517042535
5	1101221	12.663184515	2.182658338
6	12121201	14.846426528	2.183242013
7	211110211	17.447366173	2.600939645
8	10022220021	20.961053053	3.513686880
9	111022121001	23.150109979	2.189056926
10	1222021101011	25.333347835	2.183237856



$$r = (\log_3 y^z) - (\log_3 y^{z-1}), \text{ for } y = (4)_3$$

Quaternary

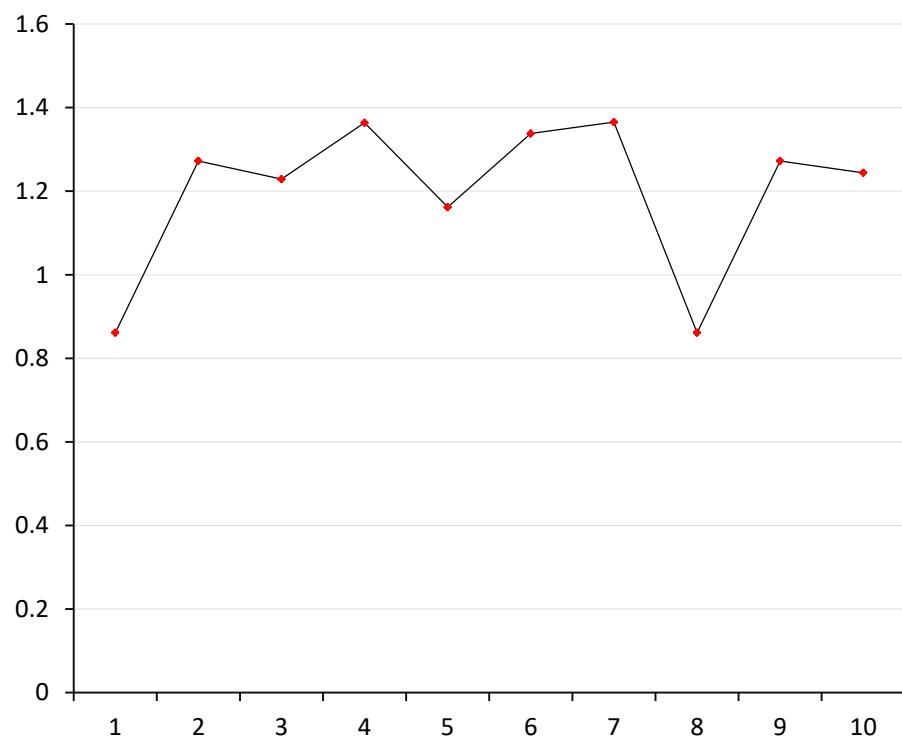
z	$y^z [x=4, b=4]$	$\log_4 y^z$	r
0	1	0	-
1	10	1.660964047	1.660964047
2	100	3.321928095	1.660964048
3	1000	4.982892142	1.660964047
4	10000	6.643856190	1.660964048
5	100000	8.304820237	1.660964047
6	1000000	9.965784285	1.660964048
7	10000000	11.626748332	1.660964047
8	100000000	13.287712380	1.660964048
9	1000000000	14.948676427	1.660964047
10	10000000000	16.609640474	1.660964047



$$r = (\log_4 y^z) - (\log_4 y^{z-1}), \text{ for } y = (4)_4$$

Quinary

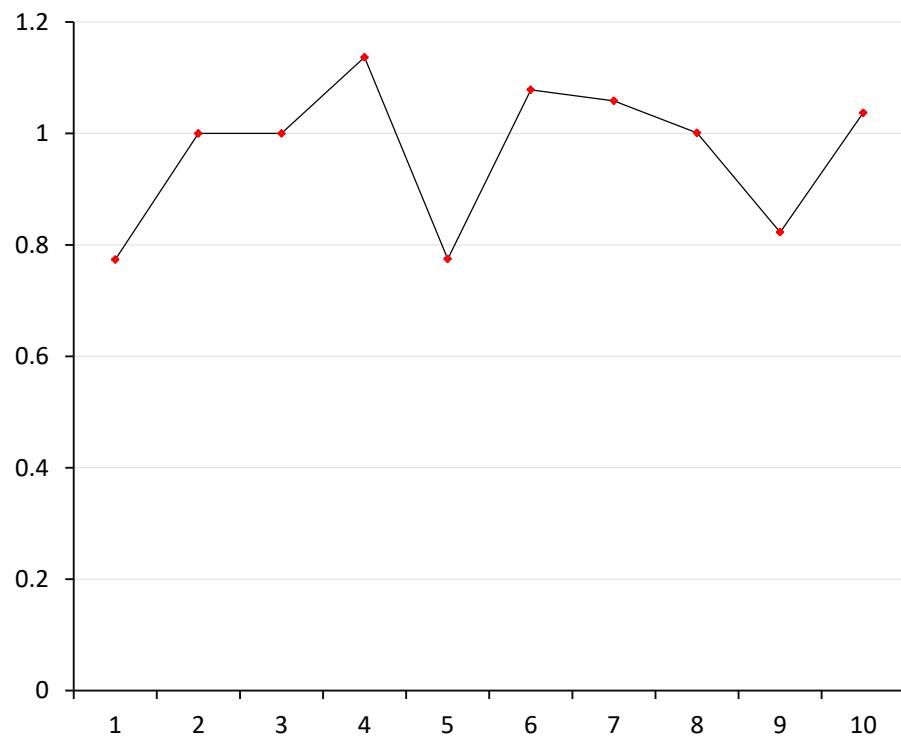
z	$y^z [x=4, b=5]$	$\log_5 y^z$	r
0	1	0	-
1	4	0.861353116	0.861353116
2	31	2.133656215	1.272303099
3	224	3.362444745	1.228788530
4	2011	4.726114211	1.363669466
5	13044	5.887821744	1.161707533
6	112341	7.225686731	1.337864987
7	1011014	8.590865319	1.365178588
8	4044121	9.452228422	0.861363103
9	31342034	10.724533429	1.272305007
10	232023301	11.968369958	1.243836529



$$r = (\log_5 y^z) - (\log_5 y^{z-1}), \text{ for } y = (4)_5$$

Senary

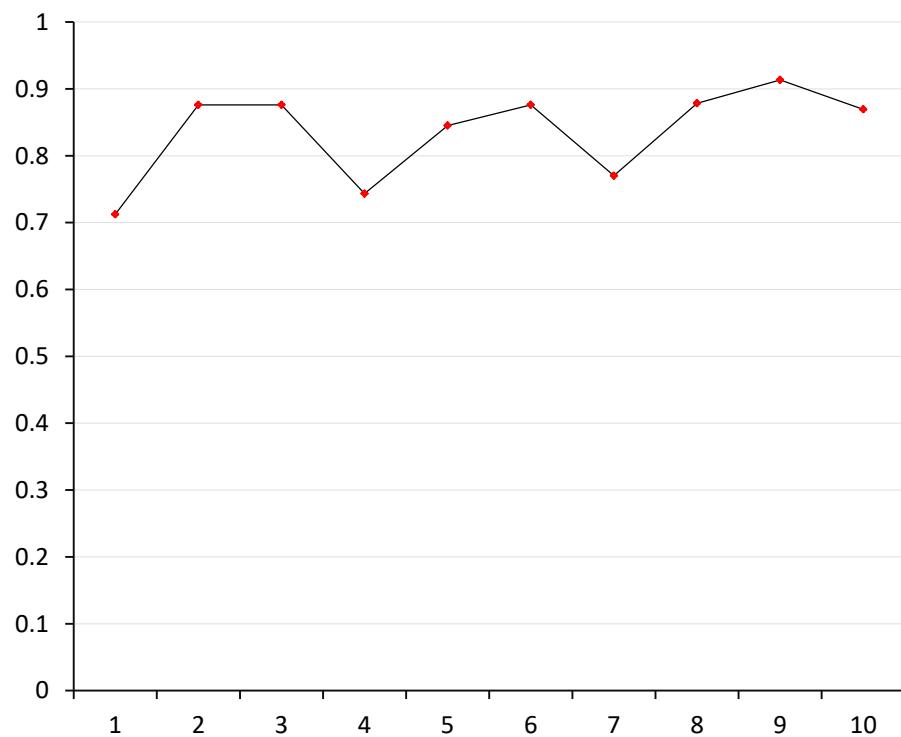
z	$y^z [x=4, b=6]$	$\log_6 y^z$	r
0	1	0	-
1	4	0.773705614	0.773705614
2	24	1.773705614	1
3	144	2.773705614	1
4	1104	3.910511063	1.136805449
5	4424	4.685226833	0.774715770
6	30544	5.763565771	1.078338938
7	203504	6.822032281	1.058466510
8	1223224	7.823036962	1.001004681
9	5341344	8.645684943	0.822647981
10	34250304	9.682776230	1.037091287



$$r = (\log_6 y^z) - (\log_6 y^{z-1}), \text{ for } y = (4)_6$$

Septenary

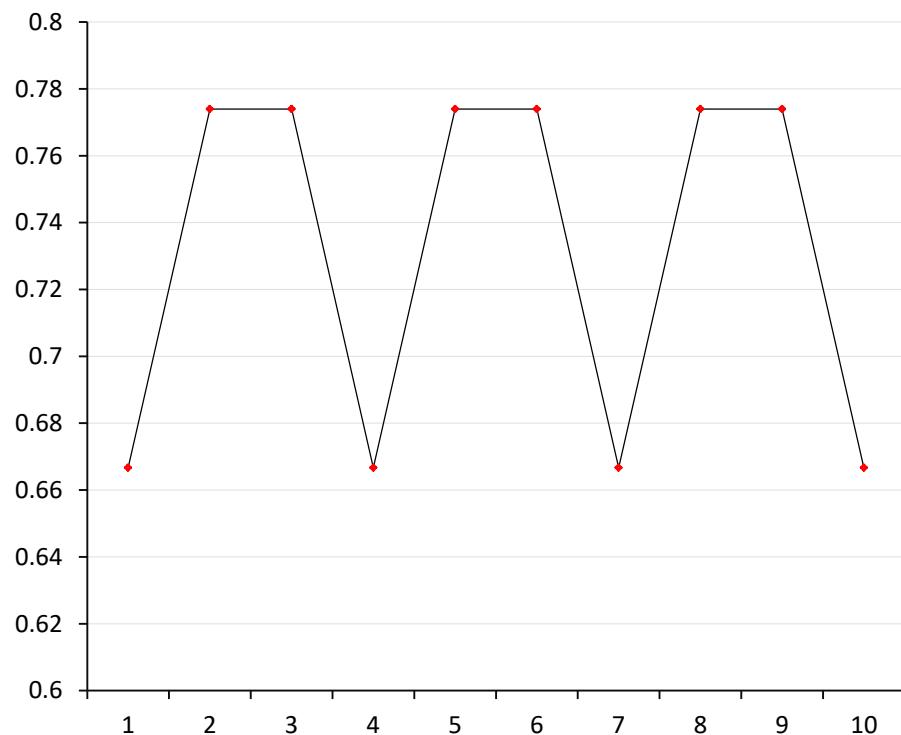
z	$y^z [x=4, b=7]$	$\log_7 y^z$	r
0	1	0	-
1	4	0.712414374	0.712414374
2	22	1.588481593	0.876067219
3	121	2.464548812	0.876067219
4	514	3.207868189	0.743319377
5	2662	4.053030405	0.845162216
6	14641	4.929097623	0.876067218
7	65524	5.699220888	0.770123265
8	362032	6.577635607	0.878414719
9	2141161	7.491023555	0.913387948
10	11625034	8.360443283	0.869419728



$$r = (\log_7 y^z) - (\log_7 y^{z-1}), \text{ for } y = (4)_7$$

Octal

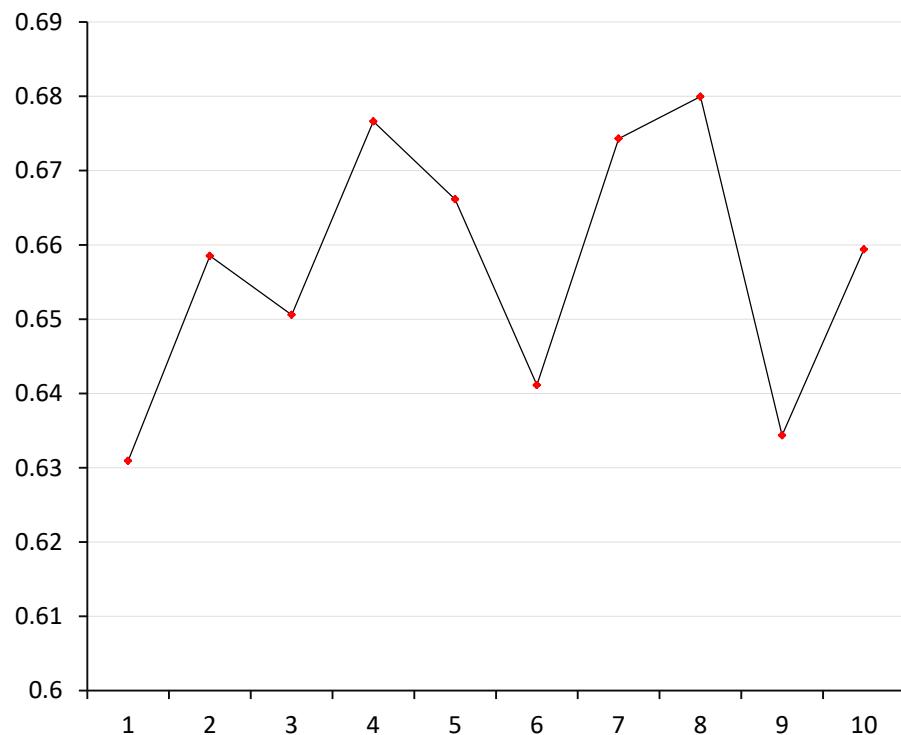
z	$y^z [x=4, b=8]$	$\log_8 y^z$	r
0	1	0	-
1	4	0.666666667	0.666666667
2	20	1.440642698	0.773976031
3	100	2.214618730	0.773976032
4	400	2.881285397	0.666666667
5	2000	3.655261428	0.773976031
6	10000	4.429237460	0.773976032
7	40000	5.095904127	0.666666667
8	200000	5.869880158	0.773976031
9	1000000	6.643856190	0.773976032
10	4000000	7.310522856	0.666666666



$$r = (\log_8 y^z) - (\log_8 y^{z-1}), \text{ for } y = (4)_8$$

Nonary

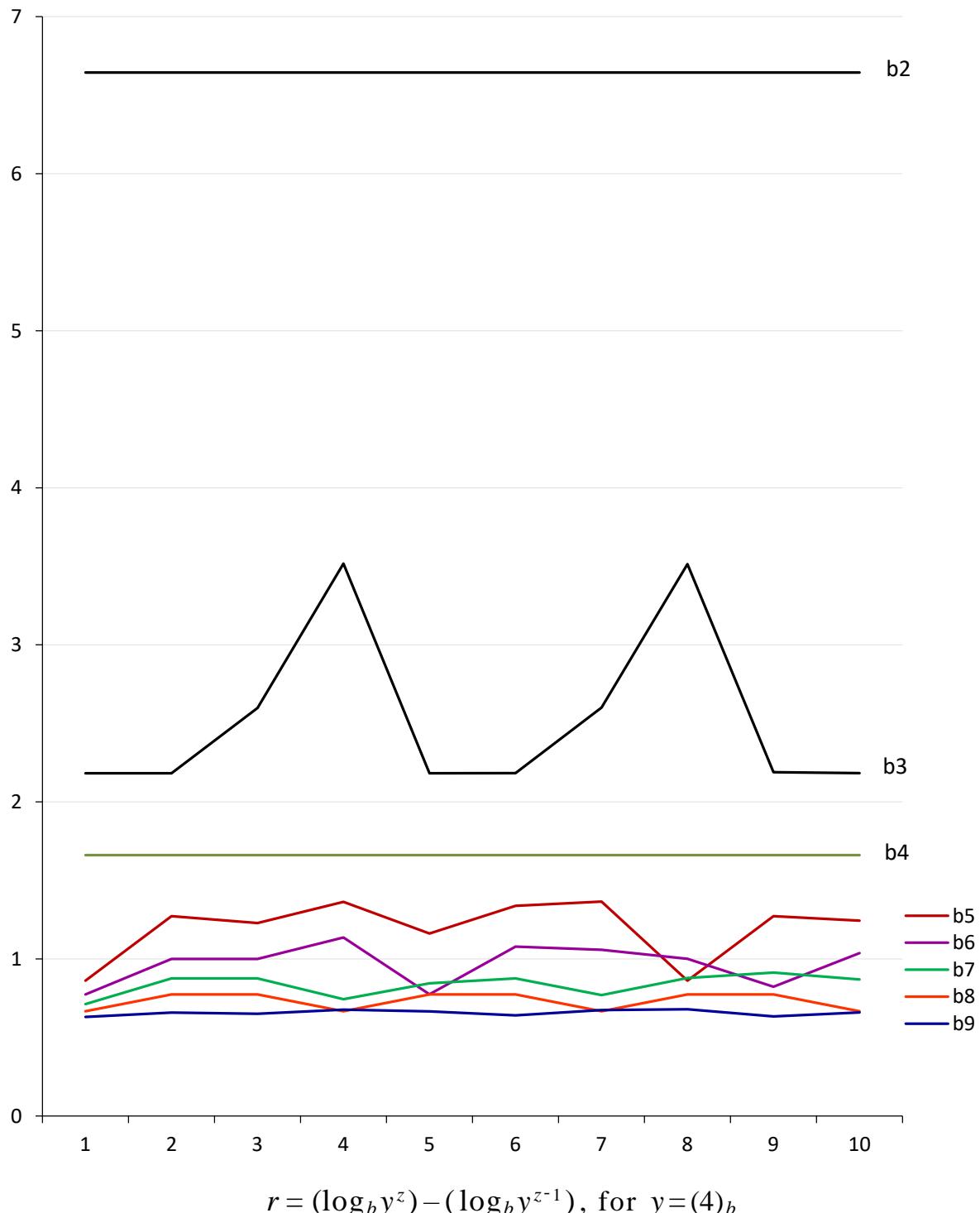
z	$y^z [x=4, b=9]$	$\log_9 y^z$	r
0	1	0	-
1	4	0.630929754	0.630929754
2	17	1.289450962	0.658521208
3	71	1.940029217	0.650578255
4	314	2.616661513	0.676632296
5	1357	3.282792180	0.666130667
6	5551	3.923919958	0.641127778
7	24424	4.598219790	0.674299832
8	108807	5.278172777	0.679952987
9	438531	5.912543450	0.634370673
10	1867334	6.571937367	0.659393917

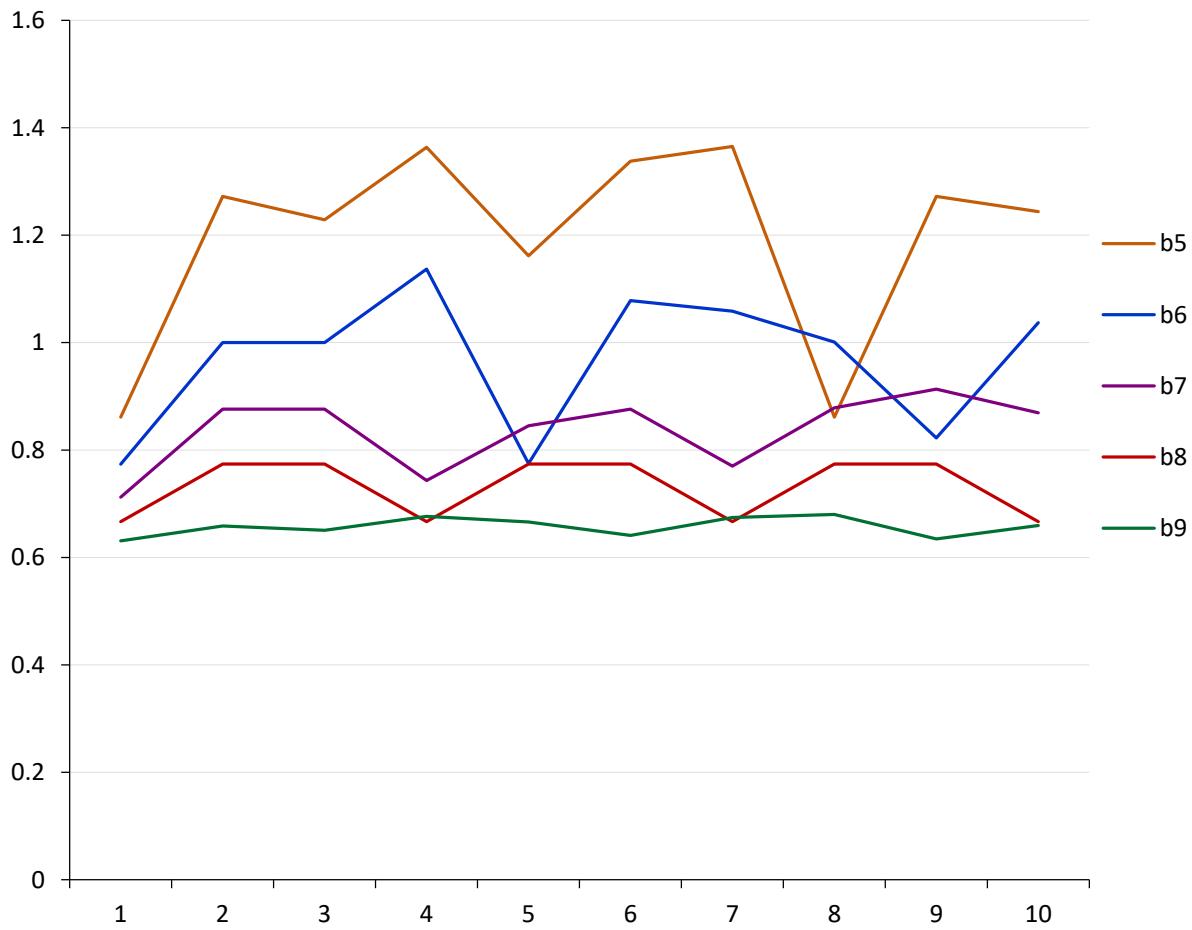


$$r = (\log_9 y^z) - (\log_9 y^{z-1}), \text{ for } y = (4)_9$$

Proportional graphs

The first graph below shows the distributions represented on pages 42-49 above with a proportional vertical axis for the full range $b=(2, \dots, 9)$. The second graph shows the relationships between the lower distributions for the range $b=(5, \dots, 9)$, with an expanded vertical scale (r):





Comments

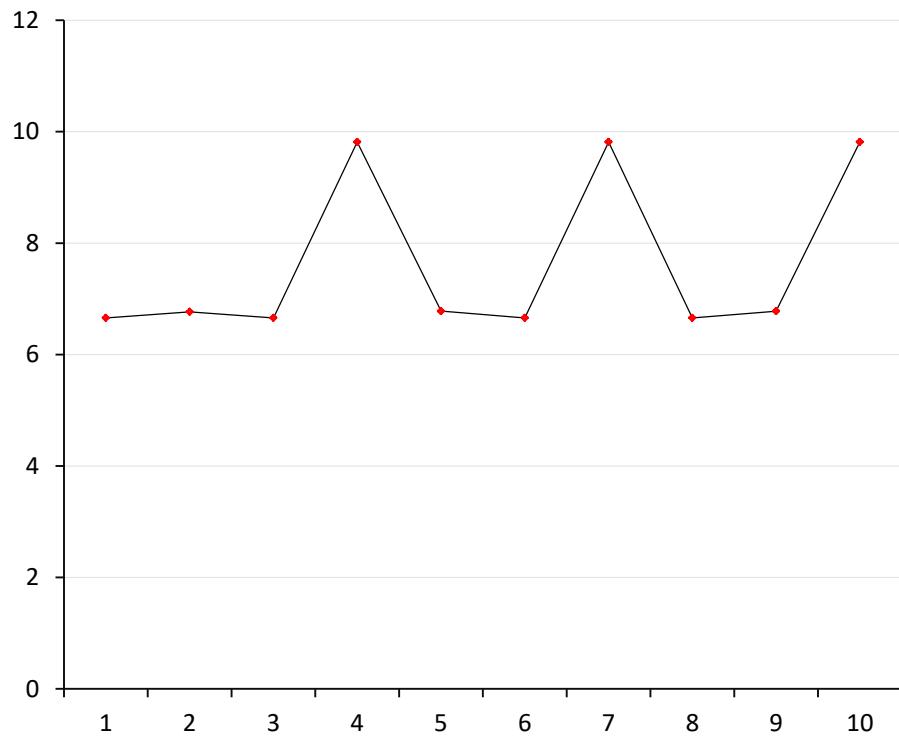
Immediately apparent from the set of distributions in the graph on p.50 is the dual appearance of lines featuring consistent values for r . The second of these (at $b=4$) is expected within the terms of the general pattern found where $x=b$. The first occurrence (at $b=2$) is in accordance with the principle noted in the *Comments* on p.31 above, i.e., where the (decimal) value of x is equal to b^2 or b^3 (or to b^n). In comparison to the equation for r where $x=b$ (i.e., $r=(\log_{10}x)^{-1}$), in this instance (for $x=b^2$), $r=4(\log_{10}x)^{-1}$. This is also the factor found in the case of the example of $x=9$, $b=3$ (see p.93 below). In comparison, in the example on p.82 below, where $x=8$, $b=2$ (i.e., $x=b^3$), the factor for r is $9(\log_{10}x)^{-1}$.

In addition to the highly unique repeating pattern found where $b=3$ in this set of distributions, there is found an equally unique and starkly regulated distribution at $b=8$ (see above p. 48), featuring three evenly spaced flattened peaks spanning $z=2$ to $z=3$; $z=5$ to $z=6$; and $z=8$ to $z=9$, the base of each peak spanning 3 values of z . There are just two values for r in this distribution: the baseline value: ≈ 0.6667 (0.6 recurring – found at $z=1, 4, 7$, & 10); and its 16% elevation: ≈ 0.774 .

$$\underline{x=5}$$

Binary

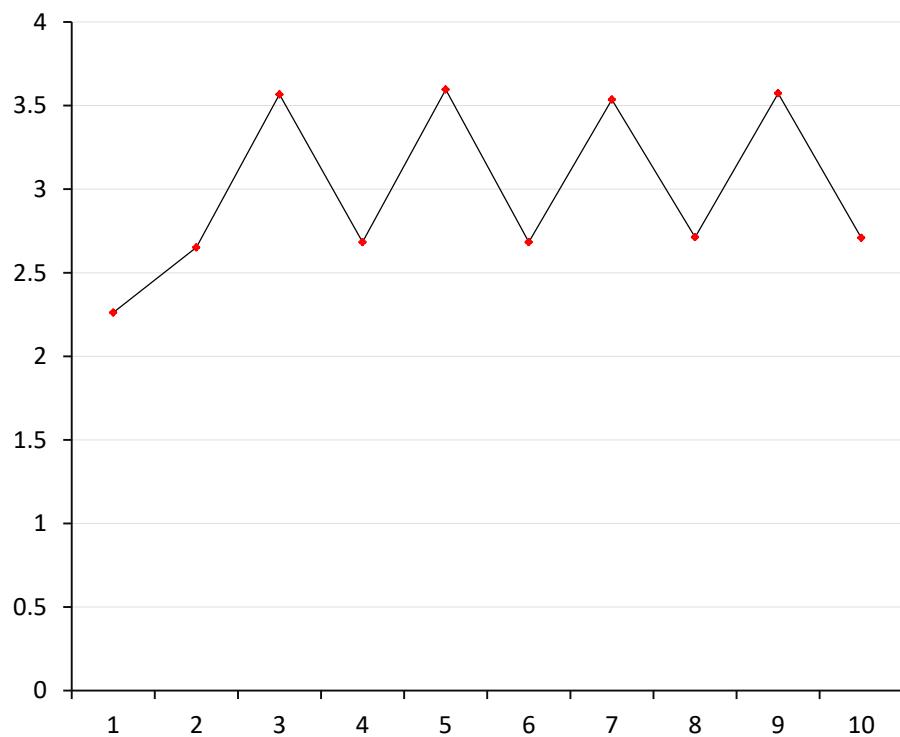
z	$y^z [x=5, b=2]$	$\log_2 y^z$	r
0	1	0	-
1	101	6.658211483	6.658211483
2	11001	13.425347051	6.767135568
3	1111101	20.083558534	6.658211483
4	1001110001	29.898953359	9.815394825
5	110000110101	36.678714012	6.779760653
6	11110100001001	43.336937036	6.658223024
7	10011000100101101	53.152435625	9.815498589
8	101111010111100001	59.810647108	6.658211483
9	111011100110101100101	66.589265838	6.778618730
10	1001010100000101111001	76.405802713	9.816536875



$$r = (\log_2 y^z) - (\log_2 y^{z-1}), \text{ for } y = (5)_2$$

Ternary

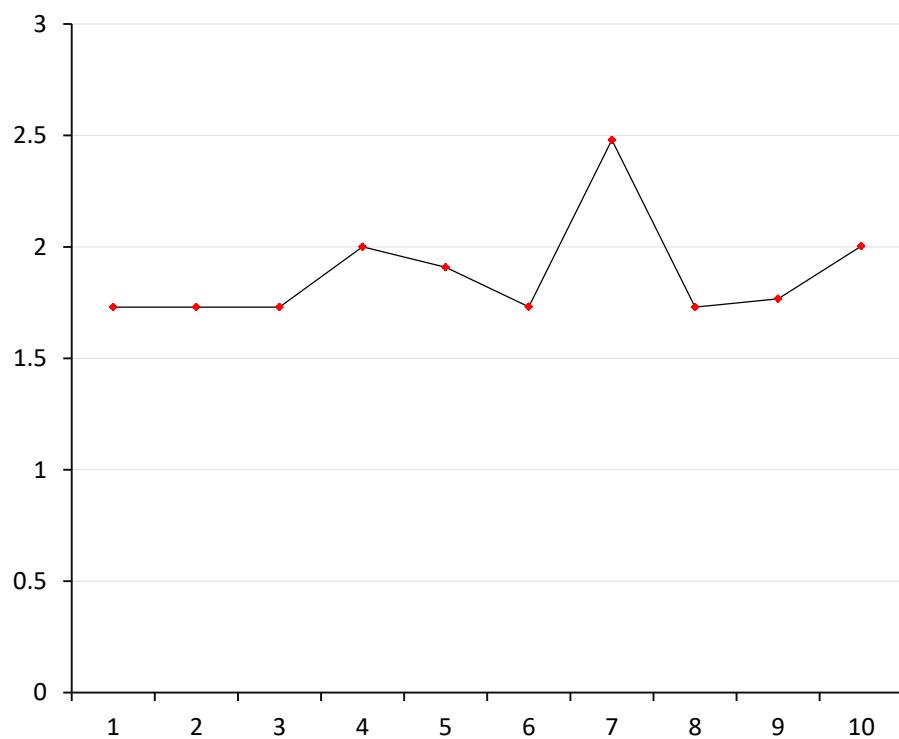
z	$y^z [x=5, b=3]$	$\log_3 y^z$	r
0	1	0	-
1	12	2.261859507	2.261859507
2	221	4.913619443	2.651759936
3	11122	8.480407969	3.566788526
4	212011	11.163531999	2.683124030
5	11021202	14.759830740	3.596298741
6	210102201	17.443009549	2.683178809
7	10222011112	20.979020007	3.536010458
8	201211211121	23.691361605	2.712341598
9	10200020011222	27.264769480	3.573407875
10	200101010220211	29.974035195	2.709265715



$$r = (\log_3 y^z) - (\log_3 y^{z-1}), \text{ for } y = (5)_3$$

Quaternary

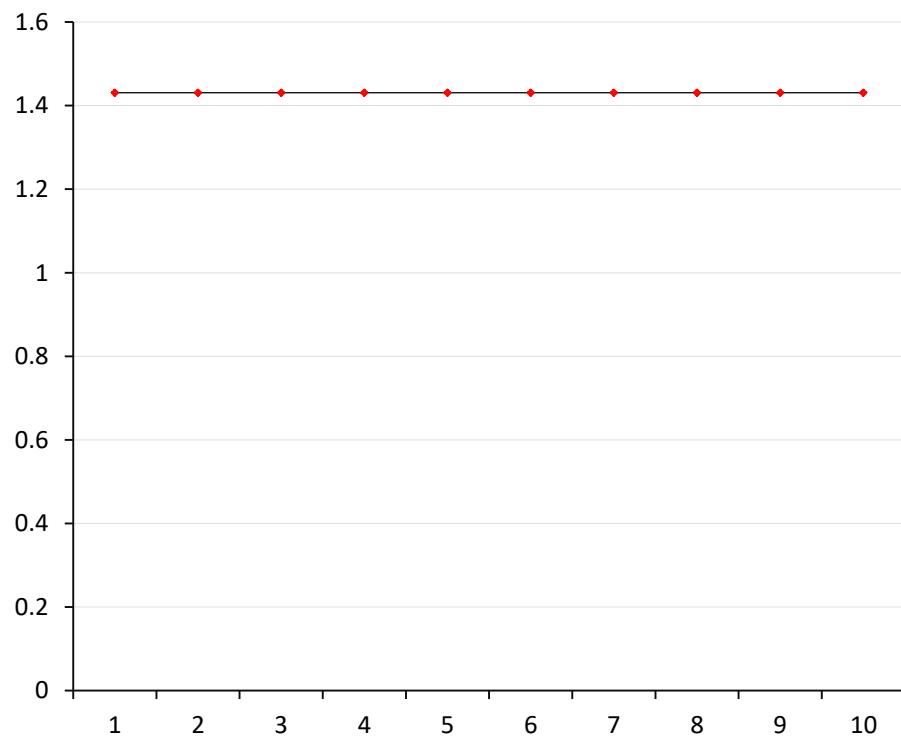
z	$y^z [x=5, b=4]$	$\log_4 y^z$	r
0	1	0	-
1	11	1.729715809	1.729715809
2	121	3.459431619	1.729715810
3	1331	5.189147428	1.729715809
4	21301	7.189316770	2.000169342
5	300311	9.098048897	1.908732127
6	3310021	10.829204470	1.731155573
7	103010231	13.309106196	2.479901726
8	1133113201	15.038822426	1.729716230
9	13130311211	16.806091030	1.767268604
10	211100023321	18.809567891	2.003476861



$$r = (\log_4 y^z) - (\log_4 y^{z-1}), \text{ for } y = (5)_4$$

Quinary

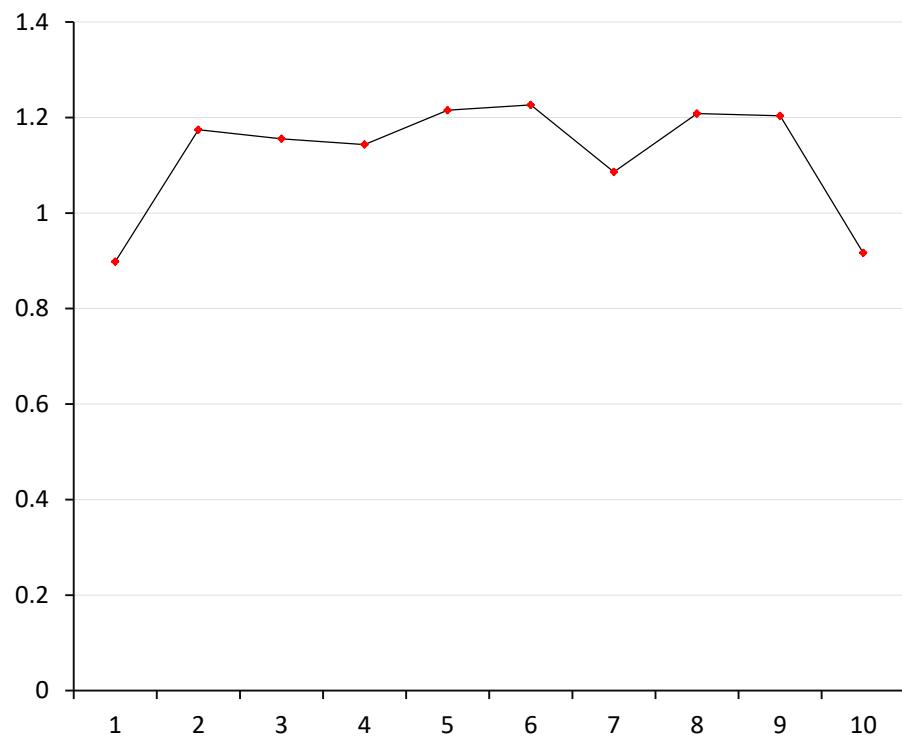
z	$y^z [x=5, b=5]$	$\log_5 y^z$	r
0	1	0	-
1	10	1.430676558	1.430676558
2	100	2.861353116	1.430676558
3	1000	4.292029674	1.430676558
4	10000	5.722706232	1.430676558
5	100000	7.153382790	1.430676558
6	1000000	8.584059348	1.430676558
7	10000000	10.014735907	1.430676559
8	100000000	11.445412465	1.430676558
9	1000000000	12.876089023	1.430676558
10	10000000000	14.306765581	1.430676558



$$r = (\log_5 y^z) - (\log_5 y^{z-1}), \text{ for } y = (5)_5$$

Senary

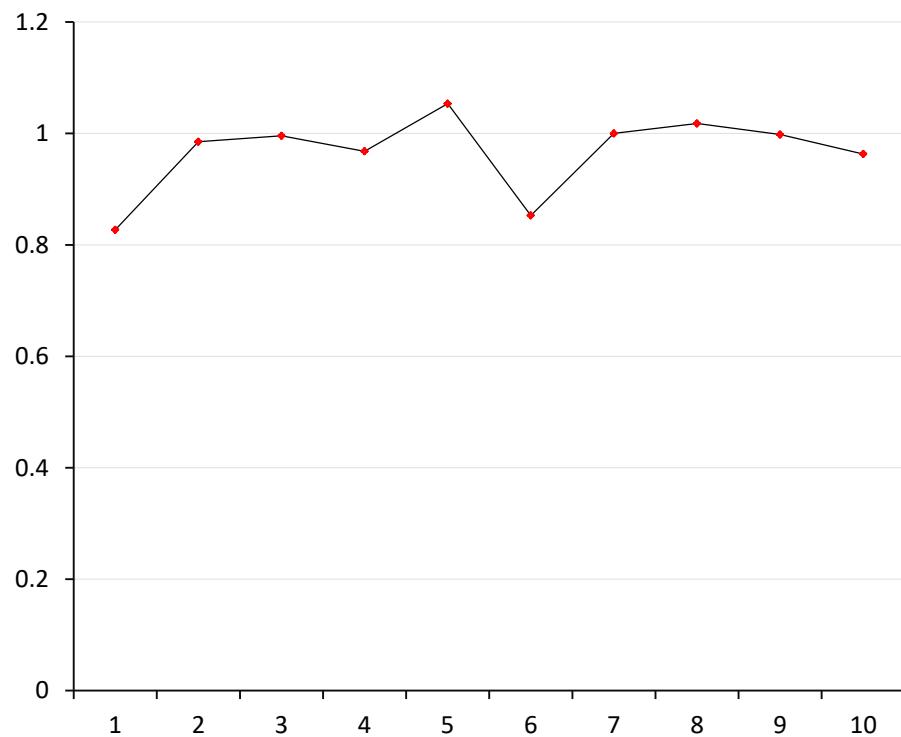
z	$y^z [x=5, b=6]$	$\log_6 y^z$	r
0	1	0	-
1	5	0.898244402	0.898244402
2	41	2.072584033	1.174339631
3	325	3.228014296	1.155430263
4	2521	4.371351770	1.143337474
5	22245	5.586616237	1.215264467
6	200201	6.812899471	1.226283234
7	1401405	7.898931807	1.086032336
8	12212241	9.107220945	1.208289138
9	105510125	10.310712903	1.203491958
10	545151121	11.227273496	0.916560593



$$r = (\log_6 y^z) - (\log_6 y^{z-1}), \text{ for } y = (5)_6$$

Septenary

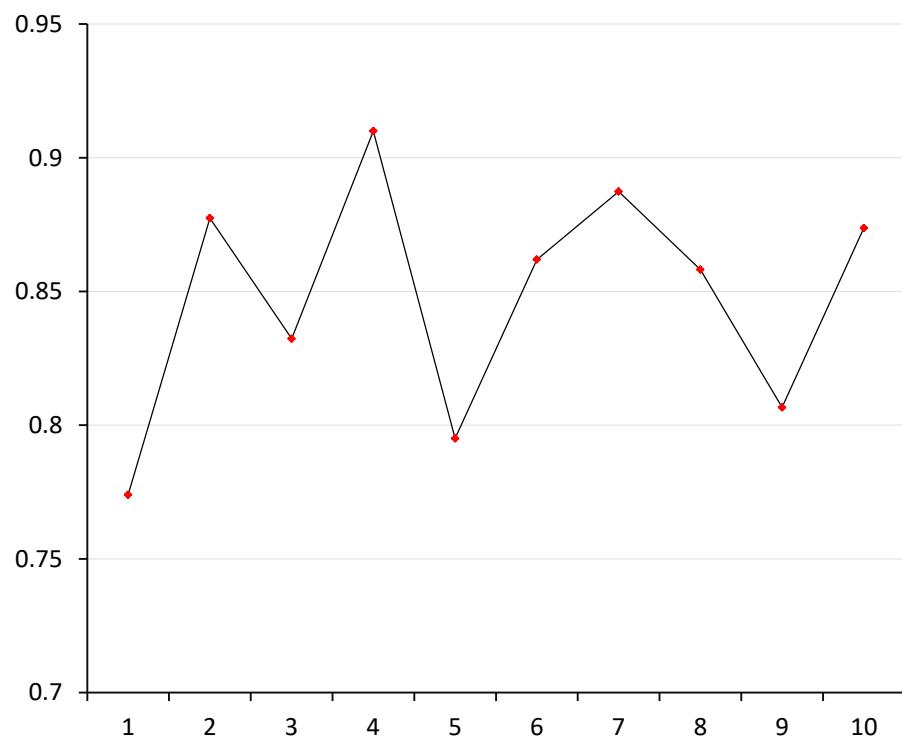
z	$y^z [x=5, b=7]$	$\log_7 y^z$	r
0	1	0	-
1	5	0.827087475	0.827087475
2	34	1.812190828	0.985103353
3	236	2.807854108	0.995663280
4	1552	3.775765137	0.967911029
5	12053	4.829138116	1.053372979
6	63361	5.681970370	0.852832254
7	443525	6.681968052	0.999997682
8	3214564	7.699842821	1.017874769
9	22413146	8.697810753	0.997967932
10	146002132	9.660842661	0.963031908



$$r = (\log_7 y^z) - (\log_7 y^{z-1}), \text{ for } y = (5)_7$$

Octal

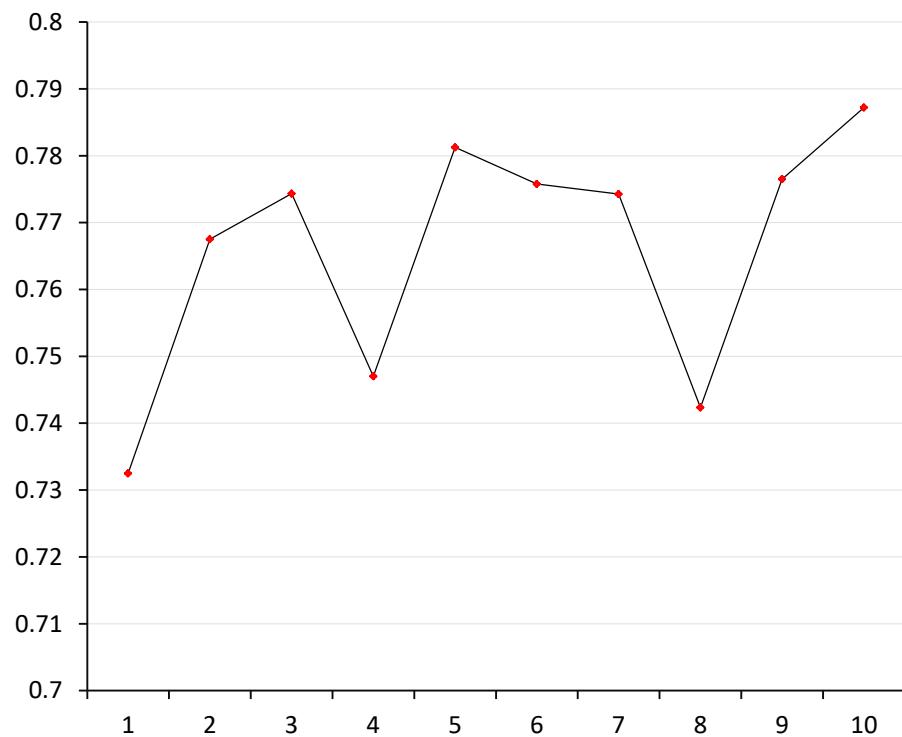
z	$y^z [x=5, b=8]$	$\log_8 y^z$	r
0	1	0	-
1	5	0.773976032	0.773976032
2	31	1.651398770	0.877422738
3	175	2.483737037	0.832338267
4	1161	3.393717419	0.909980382
5	6065	4.188763977	0.795046558
6	36411	5.050695581	0.861931604
7	230455	5.938041848	0.887346267
8	1372741	6.796209341	0.858167493
9	7346545	7.602878165	0.806668824
10	45201371	8.476621065	0.873742900



$$r = (\log_8 y^z) - (\log_8 y^{z-1}), \text{ for } y = (5)_8$$

Nonary

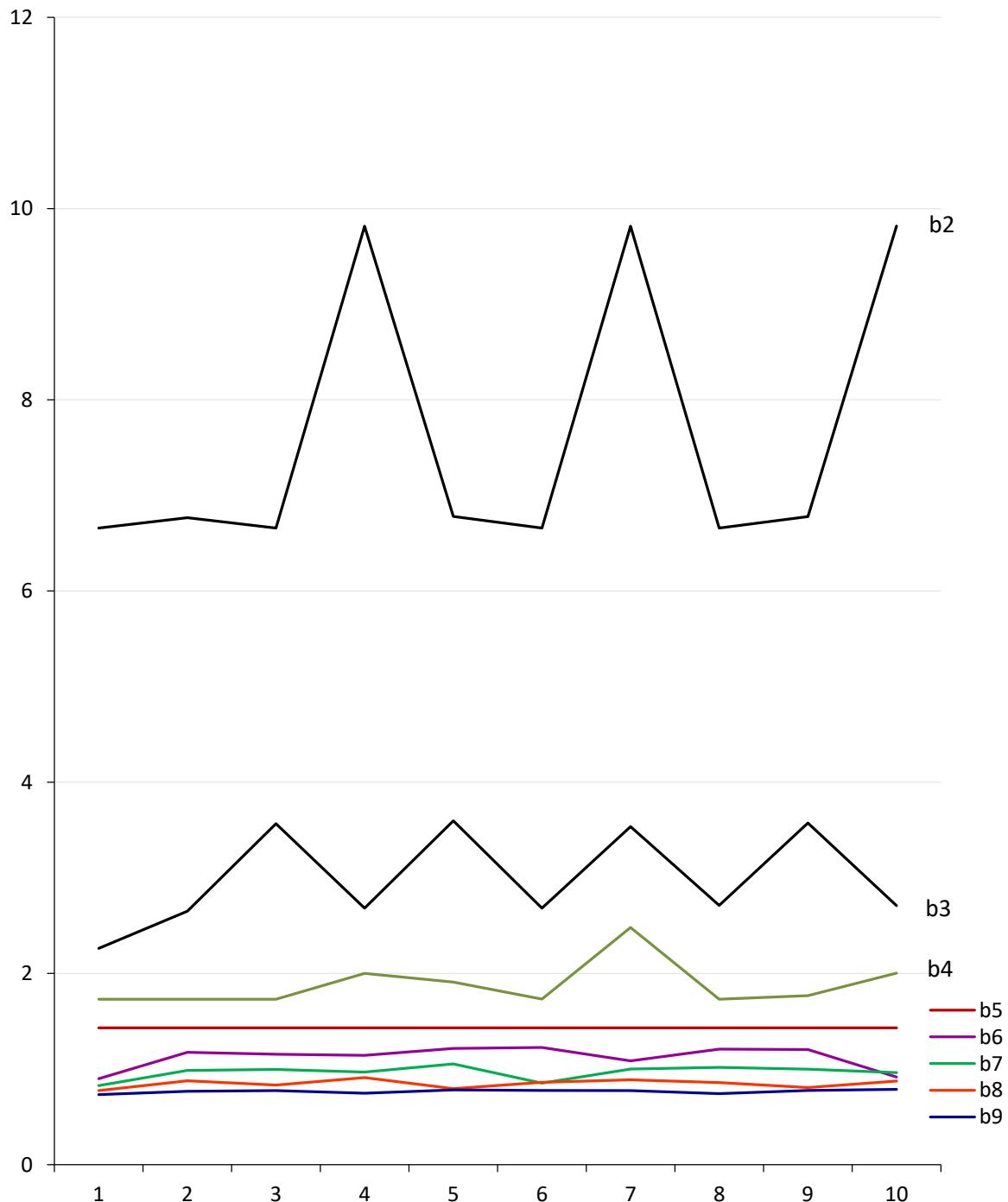
z	$y^z [x=5, b=9]$	$\log_9 y^z$	r
0	1	0	-
1	5	0.732486760	0.732486760
2	27	1.5	0.767513240
3	148	2.274329318	0.774329318
4	764	3.021342405	0.747013087
5	4252	3.802590243	0.781247838
6	23381	4.578357219	0.775766976
7	128145	5.352624322	0.774267103
8	654747	6.094963764	0.742339442
9	3606158	6.871465782	0.776502018
10	20333824	7.658660129	0.787194347



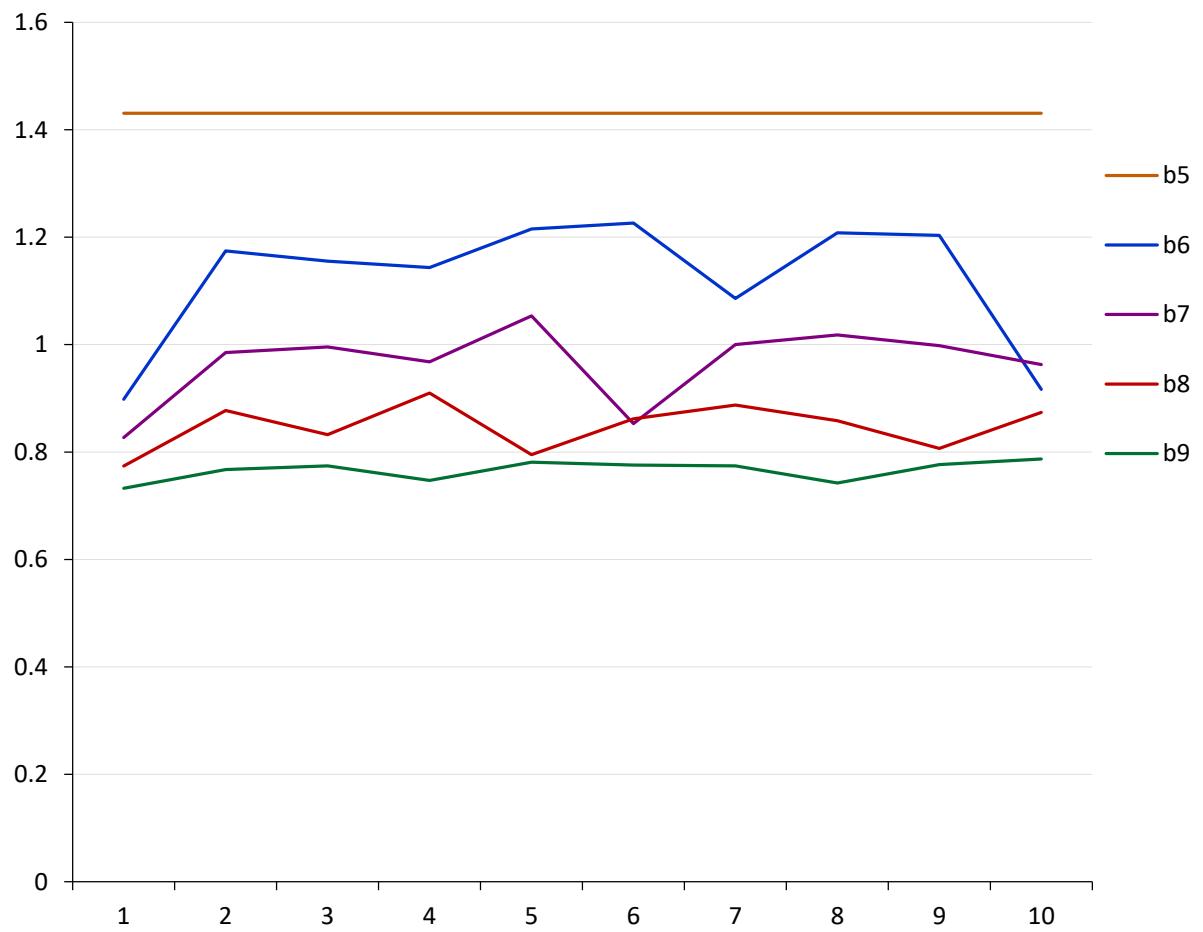
$$r = (\log_9 y^z) - (\log_9 y^{z-1}), \text{ for } y = (5)_9$$

Proportional graphs

The first graph below shows the distributions represented on pages 52-59 above with a proportional vertical axis for the full range $b=(2, \dots, 9)$. The second graph shows the relationships between the lower distributions for the range $b=(5, \dots, 9)$, with an expanded vertical scale (r):



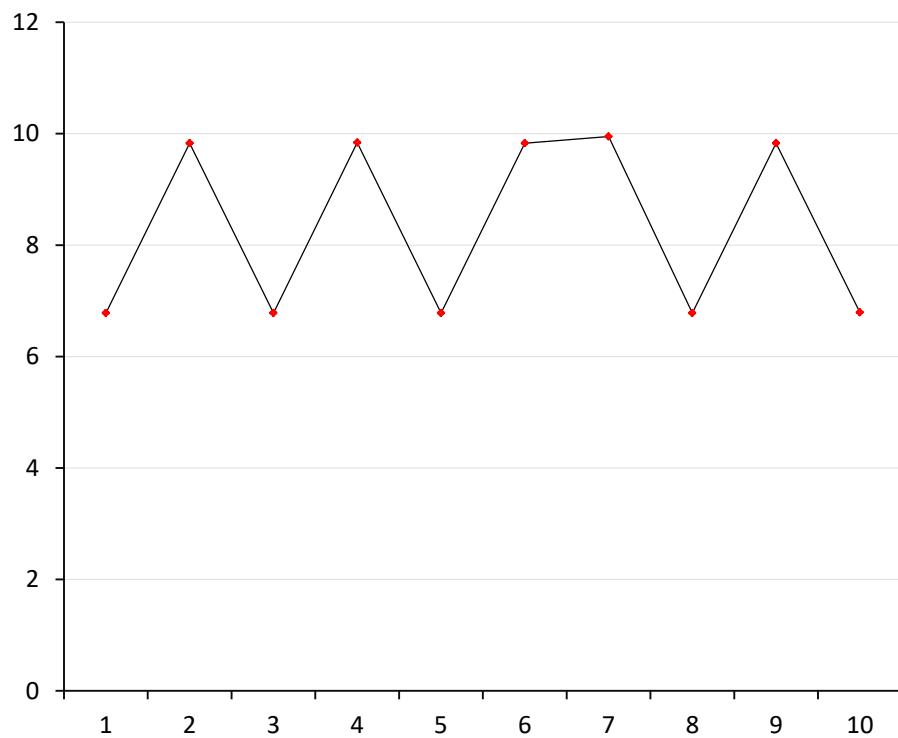
$$r = (\log_b y^z) - (\log_b y^{z-1}), \text{ for } y=(5)_b$$



$$\underline{x=6}$$

Binary

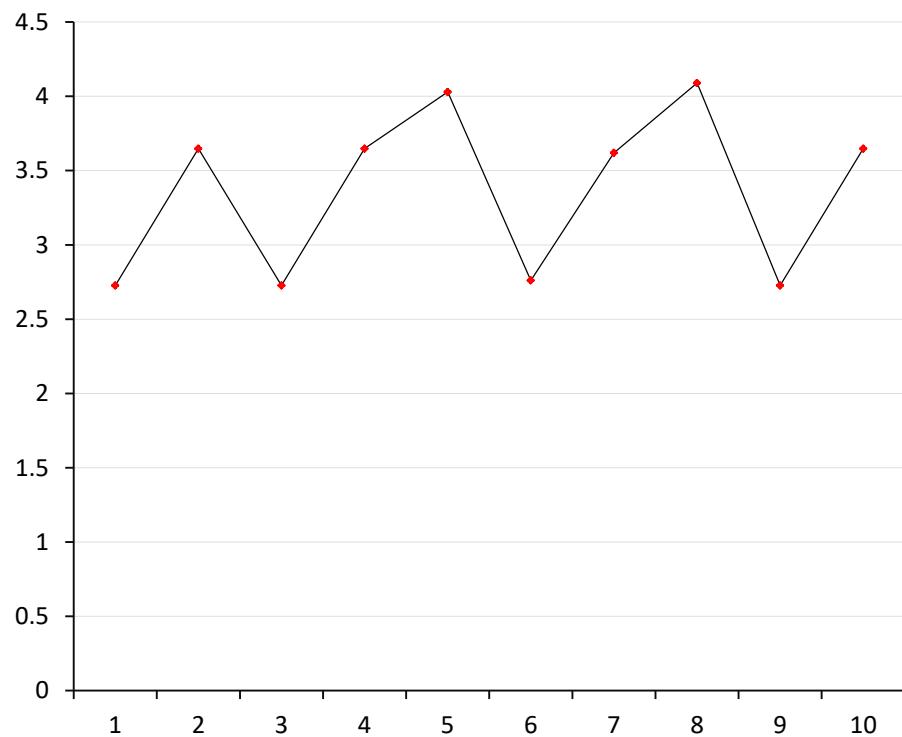
z	$y^z [x=6, b=2]$	$\log_2 y^z$	r
0	1	0	-
1	110	6.781359714	6.781359714
2	100100	16.611082449	9.829722735
3	11011000	23.392442162	6.781359713
4	10100010000	33.233637670	9.841195508
5	1111001100000	40.014997384	6.781359714
6	1011011001000000	49.844720119	9.829722735
7	1000100010110000000	59.794849985	9.950129866
8	11001101000100000000	66.576209815	6.781359830
9	100110011100011000000000	76.405932435	9.829722620
10	11100110101010010000000000	83.198776359	6.792843924



$$r = (\log_2 y^z) - (\log_2 y^{z-1}), \text{ for } y = (6)_2$$

Ternary

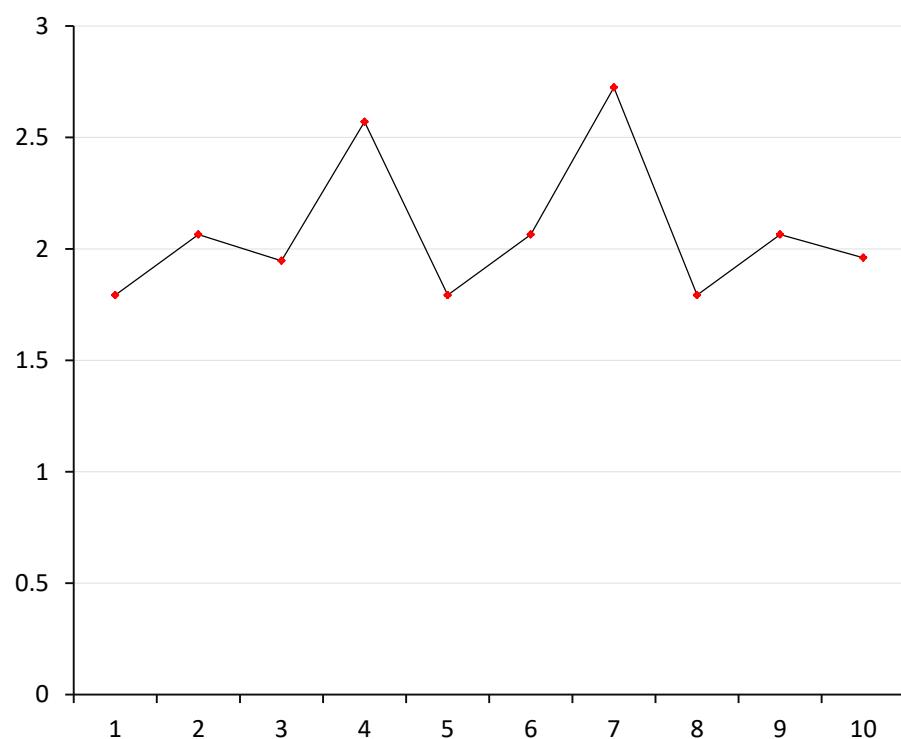
z	$y^z [x=6, b=3]$	$\log_3 y^z$	r
0	1	0	-
1	20	2.726833028	2.726833028
2	1100	6.374464887	3.647631859
3	22000	9.101297915	2.726833028
4	1210000	12.748929774	3.647631859
5	101200000	16.778084047	4.029154273
6	2101000000	19.538903288	2.760819241
7	112020000000	23.158254760	3.619351472
8	10011100000000	27.247752371	4.089497611
9	200222000000000	29.974585399	2.726833028
10	1101221000000000	33.622217258	3.647631859



$$r = (\log_3 y^z) - (\log_3 y^{z-1}), \text{ for } y = (6)_3$$

Quaternary

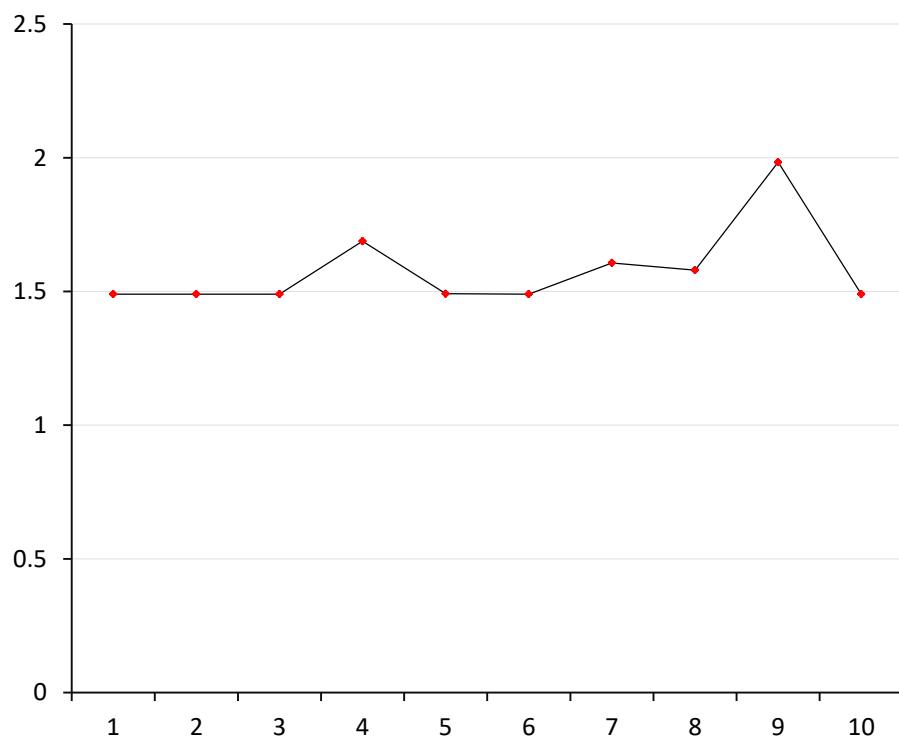
z	$y^z [x=6, b=4]$	$\log_4 y^z$	r
0	1	0	-
1	12	1.792481250	1.792481250
2	210	3.857122759	2.064641509
3	3120	5.803665157	1.946542398
4	110100	8.374227472	2.570562315
5	1321200	10.166708722	1.792481250
6	23121000	12.231350231	2.064641509
7	1010112000	14.955934060	2.724583829
8	12122010000	16.748454943	1.792520883
9	212130120000	18.813079263	2.064624320
10	3212222100000	20.773354393	1.960275130



$$r = (\log_4 y^z) - (\log_4 y^{z-1}), \text{ for } y = (6)_4$$

Quinary

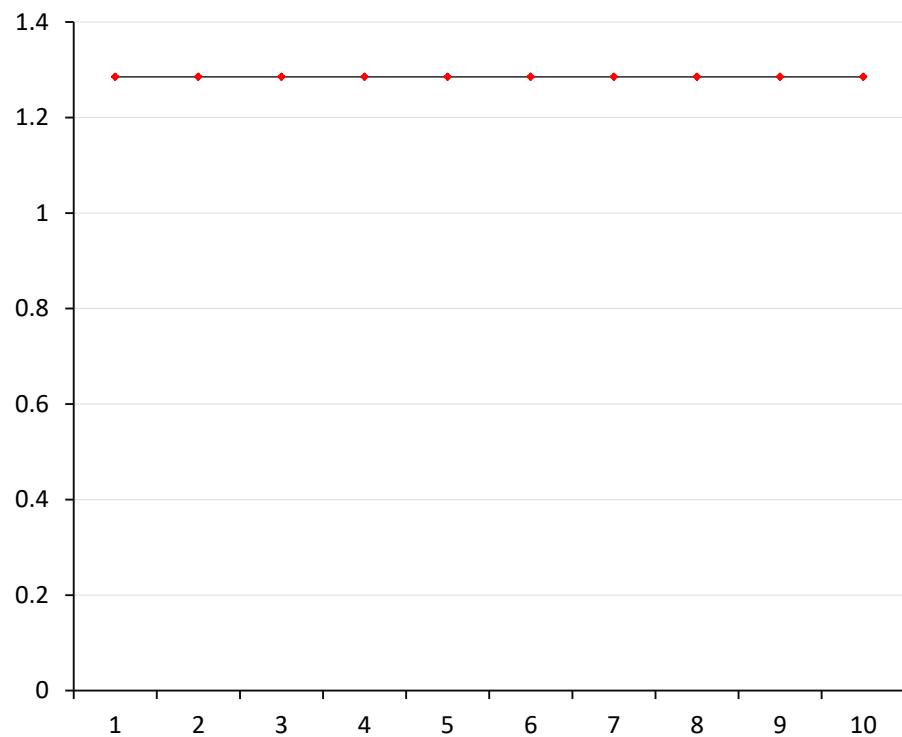
z	$y^z [x=6, b=5]$	$\log_5 y^z$	r
0	1	0	-
1	11	1.489896102	1.489896102
2	121	2.979792205	1.489896103
3	1331	4.469688307	1.489896102
4	20141	6.157747833	1.688059526
5	222101	7.649184487	1.491436654
6	2443111	9.139080589	1.489896102
7	32424221	10.745624997	1.606544408
8	412221431	12.325465346	1.579840349
9	10034441241	14.308901859	1.983436513
10	110434404201	15.799110583	1.490208724



$$r = (\log_5 y^z) - (\log_5 y^{z-1}), \text{ for } y = (6)_5$$

Senary

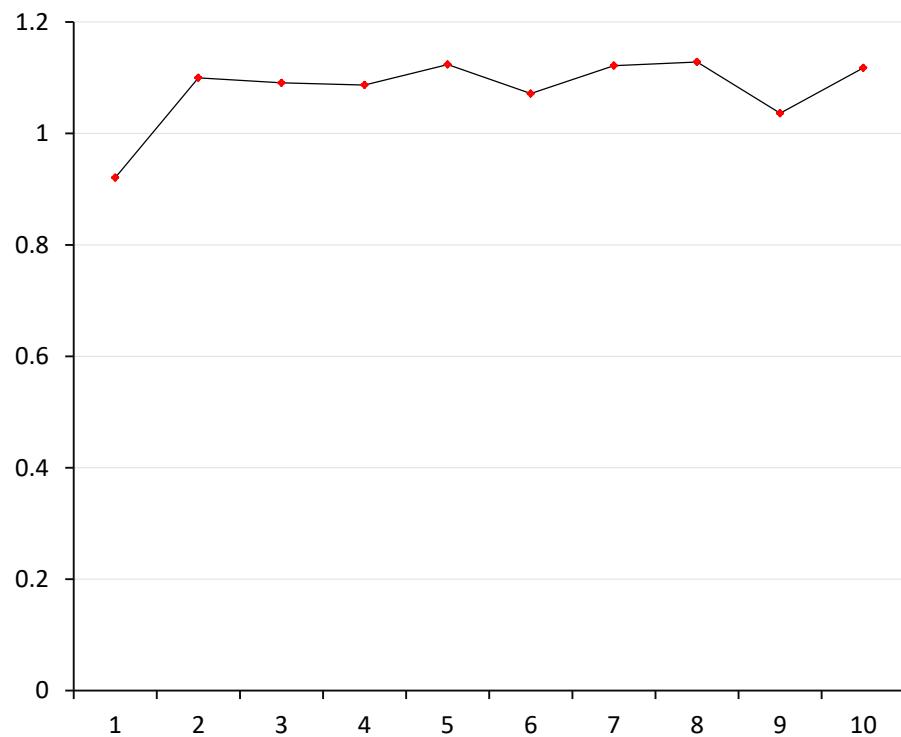
z	$y^z [x=6, b=6]$	$\log_6 y^z$	r
0	1	0	-
1	10	1.285097209	1.285097209
2	100	2.570194418	1.285097209
3	1000	3.855291627	1.285097209
4	10000	5.140388836	1.285097209
5	100000	6.425486045	1.285097209
6	1000000	7.710583254	1.285097209
7	10000000	8.995680463	1.285097209
8	100000000	10.280777672	1.285097209
9	1000000000	11.565874880	1.285097208
10	10000000000	12.850972089	1.285097209



$$r = (\log_6 y^z) - (\log_6 y^{z-1}), \text{ for } y = (6)_6$$

Septenary

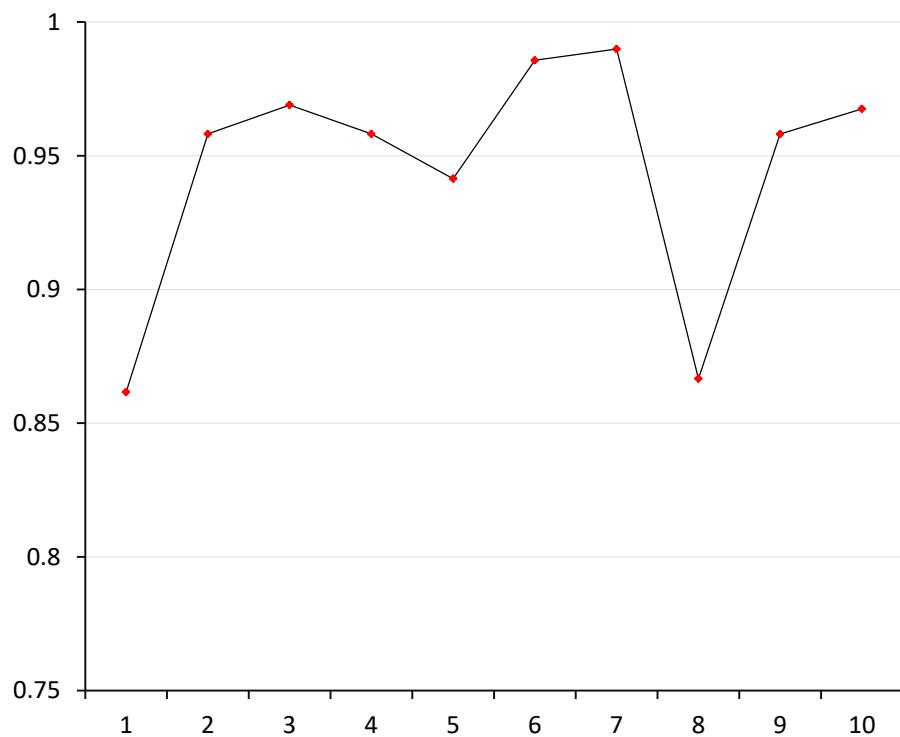
z	$y^z [x=6, b=7]$	$\log_7 y^z$	r
0	1	0	-
1	6	0.920782221	0.920782221
2	51	2.020558675	1.099776454
3	426	3.111366344	1.090807669
4	3531	4.198208432	1.086842088
5	31446	5.321945144	1.123736712
6	253011	6.393506016	1.071560872
7	2244066	7.515146557	1.121640541
8	20163561	8.643455419	1.128308862
9	151442016	9.679641882	1.036186463
10	1332645141	10.797225988	1.117584106



$$r = (\log_7 y^z) - (\log_7 y^{z-1}), \text{ for } y = (6)_7$$

Octal

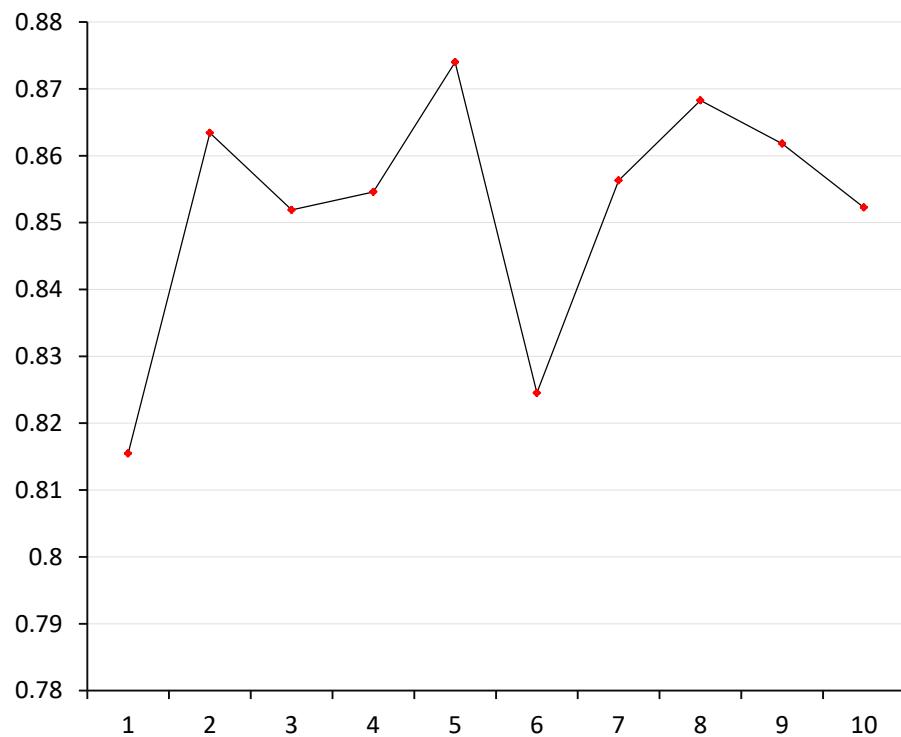
z	$y^z [x=6, b=8]$	$\log_8 y^z$	r
0	1	0	-
1	6	0.861654167	0.861654167
2	44	1.819810540	0.958156373
3	330	2.788774071	0.968963531
4	2420	3.746930444	0.958156373
5	17140	4.688359830	0.941429386
6	133100	5.674050349	0.985690519
7	1042600	6.663918111	0.989867762
8	6320400	7.530528145	0.866610034
9	46343000	8.488616035	0.958087890
10	346522000	9.456123903	0.967507868



$$r = (\log_8 y^z) - (\log_8 y^{z-1}), \text{ for } y = (6)_8$$

Nonary

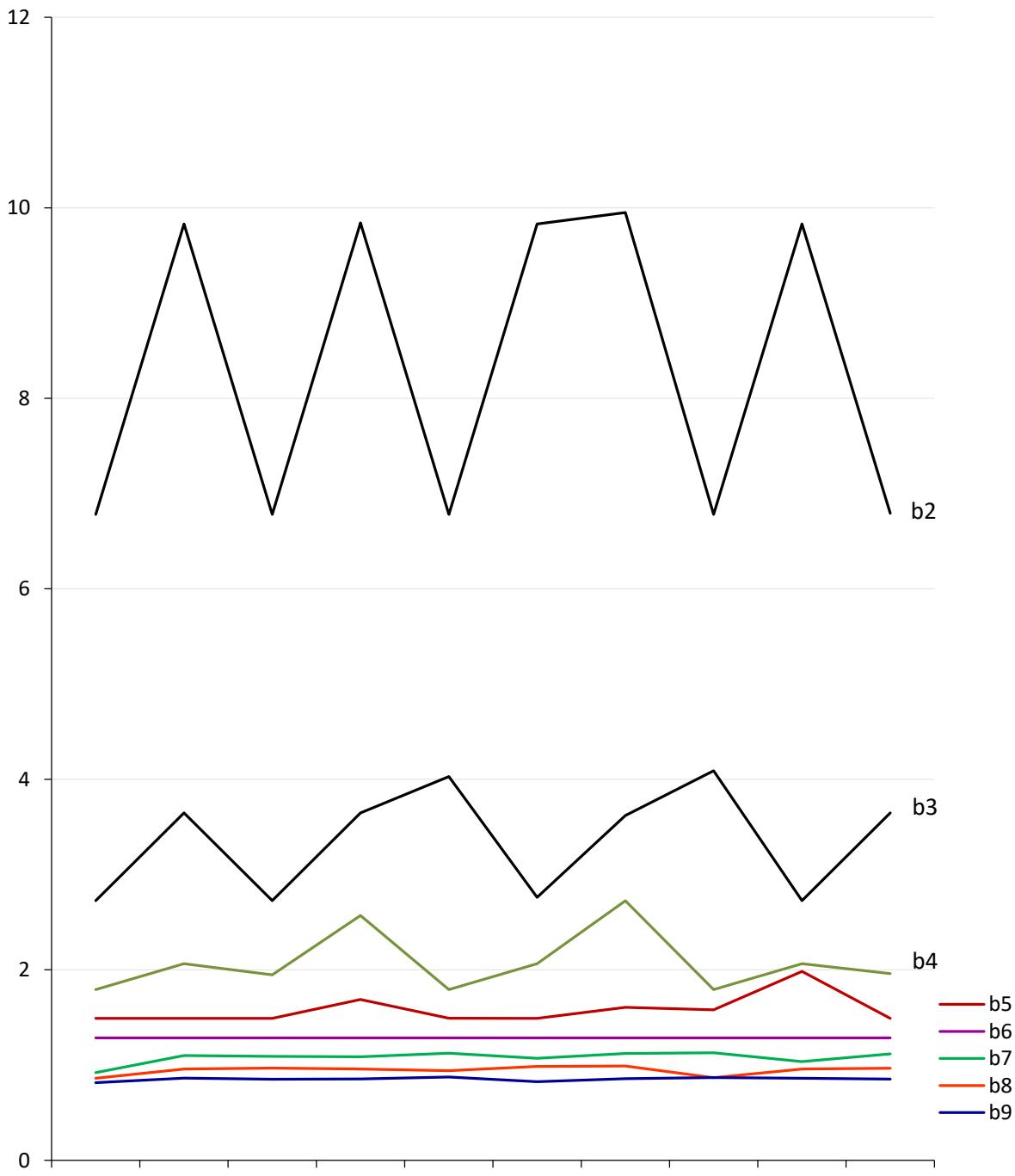
z	$y^z [x=6, b=9]$	$\log_9 y^z$	r
0	1	0	-
1	6	0.815464877	0.815464877
2	40	1.678881391	0.863416514
3	260	2.530775274	0.851893883
4	1700	3.385354236	0.854578962
5	11600	4.259355404	0.874001168
6	71000	5.083884129	0.824528725
7	466000	5.940194301	0.856310172
8	3140000	6.808468061	0.868273760
9	20860000	7.670287408	0.861819347
10	135700000	8.522550366	0.852262958



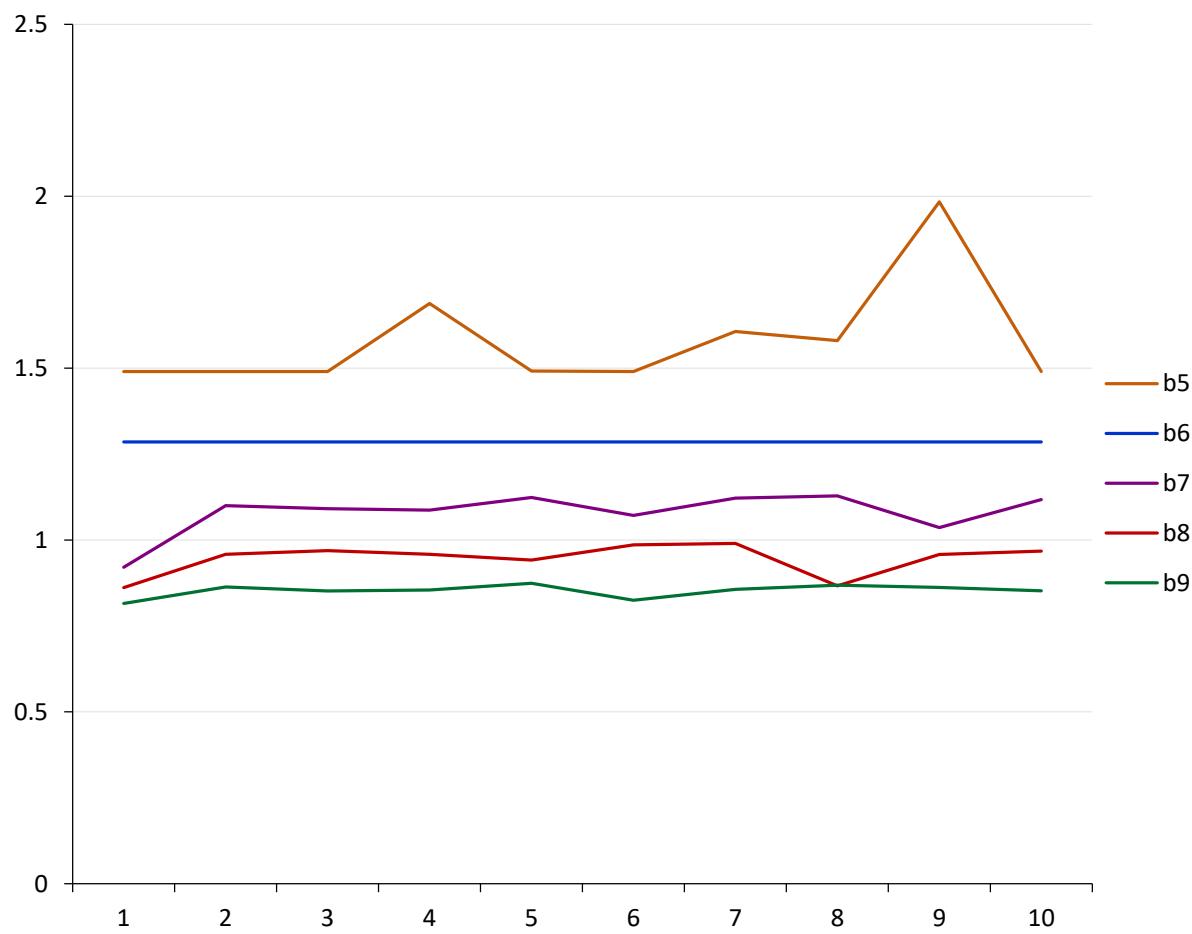
$$r = (\log_9 y^z) - (\log_9 y^{z-1}), \text{ for } y = (6)_9$$

Proportional graphs

The first graph below shows the distributions represented on pages 62-69 above with a proportional vertical axis for the full range $b=(2, \dots, 9)$. The second graph shows the relationships between the lower distributions for the range $b=(5, \dots, 9)$, with an expanded vertical scale (r):



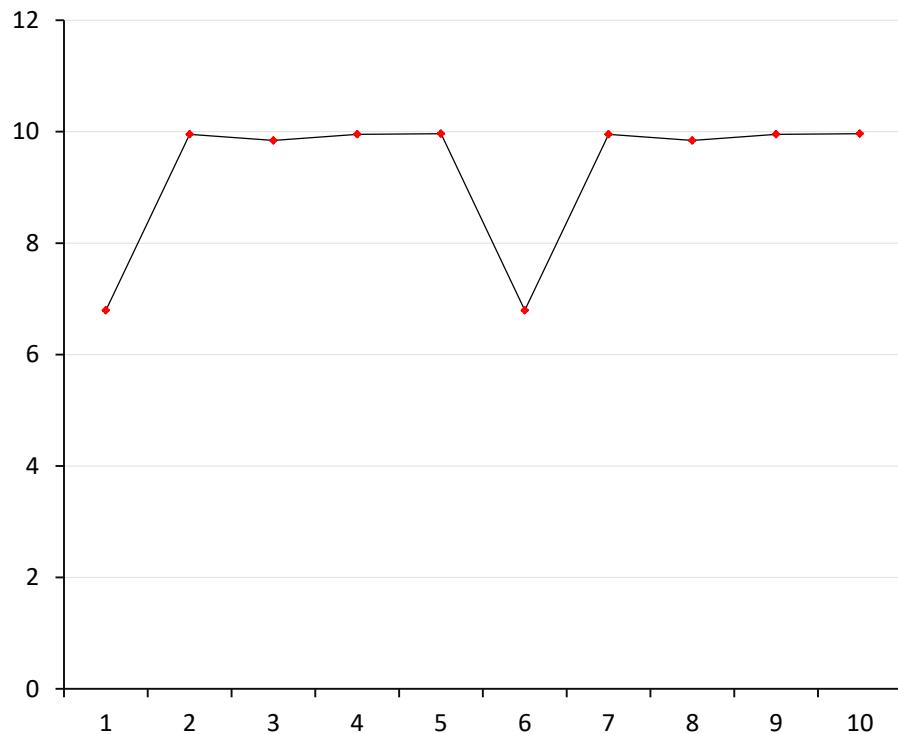
$$r = (\log_b y^z) - (\log_b y^{z-1}), \text{ for } y=(6)_b$$



$$\underline{x=7}$$

Binary

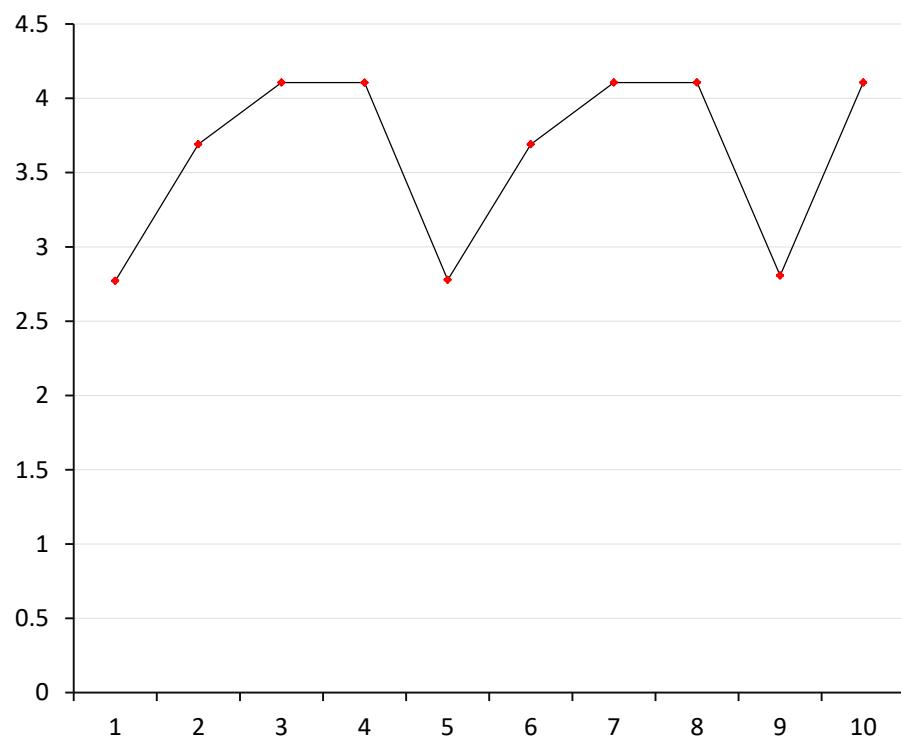
z	$y^z[x=7, b=2]$	$\log_2 y^z$	r
0	1	0	-
1	111	6.794415866	6.794415866
2	110001	16.747157114	9.952741248
3	101010111	26.589924471	9.842767357
4	100101100001	36.542666872	9.952742401
5	100000110100111	46.506994917	9.964328045
6	11100101110010001	53.301422336	6.794427419
7	11001001000011110111	63.254268606	9.952846270
8	1010111111011011000001	73.096932084	9.842663478
9	1001100111101111101000111	83.049788625	9.952856541
10	10000110101100011101011110001	93.014002541	9.964213916



$$r = (\log_2 y^z) - (\log_2 y^{z-1}), \text{ for } y = (7)_2$$

Ternary

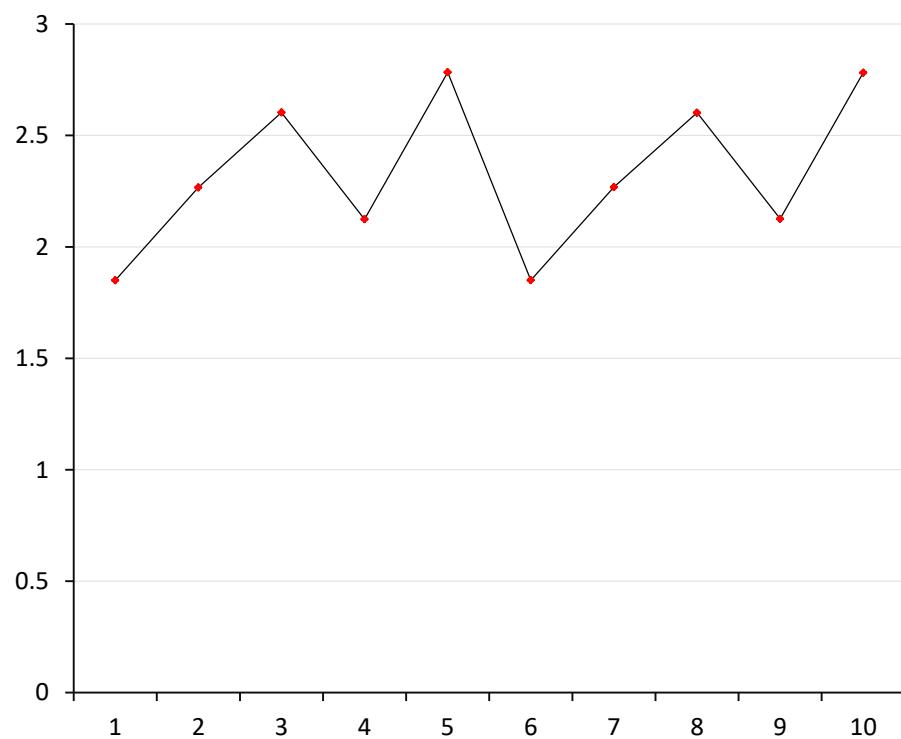
z	$y^z [x=7, b=3]$	$\log_3 y^z$	r
0	1	0	-
1	21	2.771243749	2.771243749
2	1211	6.461971905	3.690728156
3	110201	10.567933173	4.105961268
4	10021221	14.673252492	4.105319319
5	212001111	17.451199364	2.777946872
6	12222101101	21.141682061	3.690482697
7	1112211200121	25.247643330	4.105961269
8	101211212211011	29.353604535	4.105961205
9	2210221011202001	32.160453032	2.806848497
10	201200112020012021	36.266731040	4.106278008



$$r = (\log_3 y^z) - (\log_3 y^{z-1}), \text{ for } y = (7)_3$$

Quaternary

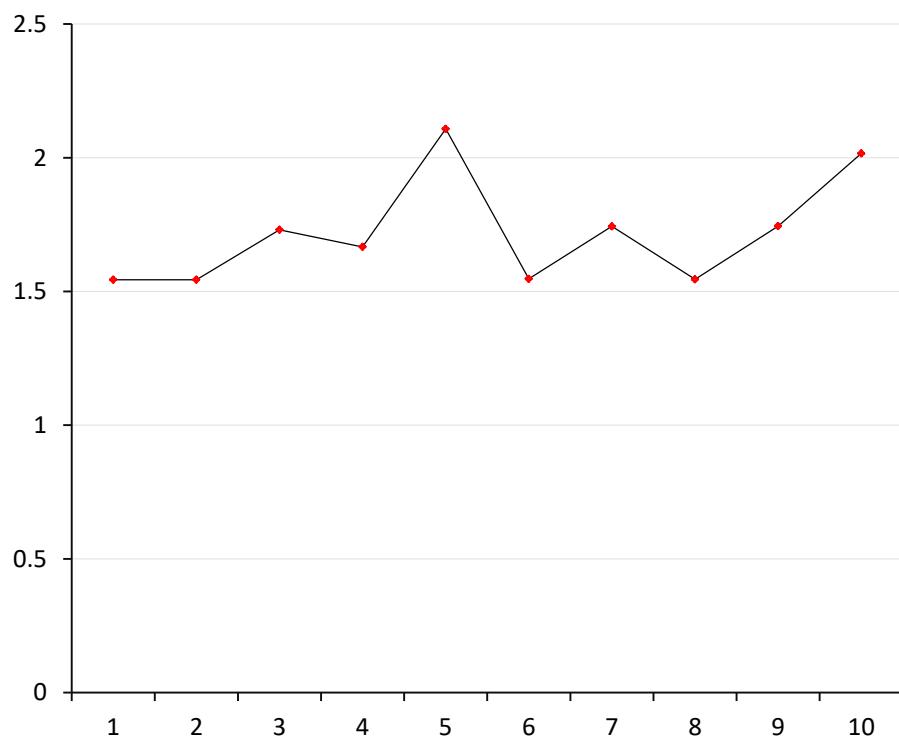
z	$y^z [x=7, b=4]$	$\log_4 y^z$	r
0	1	0	-
1	13	1.850219859	1.850219859
2	301	4.116809838	2.266589979
3	11113	6.719980355	2.603170517
4	211201	8.844128570	2.124148215
5	10012213	11.627628776	2.783500206
6	130232101	13.478254931	1.850626155
7	3021003313	15.746190310	2.267935379
8	111333123001	18.348045960	2.601855650
9	2121323331013	20.474050836	2.126004876
10	100311203223301	23.255738035	2.781687199



$$r = (\log_4 y^z) - (\log_4 y^{z-1}), \text{ for } y = (7)_4$$

Quinary

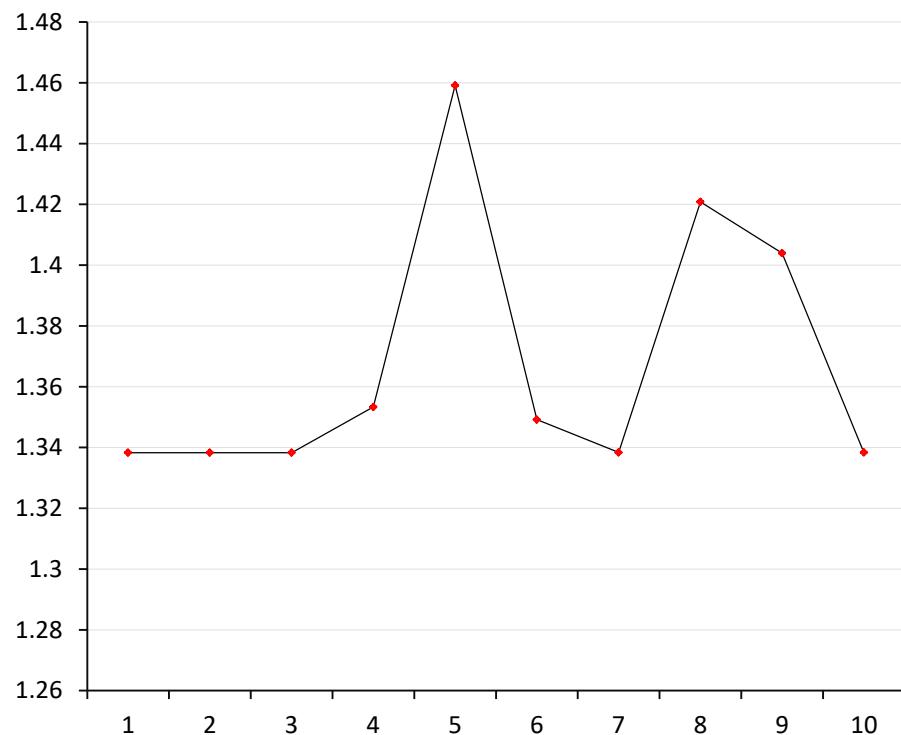
z	$y^z [x=7, b=5]$	$\log_5 y^z$	r
0	1	0	-
1	12	1.543959311	1.543959311
2	144	3.087918621	1.543959310
3	2333	4.818396666	1.730478045
4	34101	6.484923654	1.666526988
5	1014212	8.592827600	2.107903946
6	12231044	10.139867926	1.547040326
7	202323133	11.883264647	1.743396721
8	2433433201	13.428644106	1.545379459
9	40312303412	15.172950983	1.744306877
10	1034303201444	17.189075060	2.016124077



$$r = (\log_5 y^z) - (\log_5 y^{z-1}), \text{ for } y = (7)_5$$

Senary

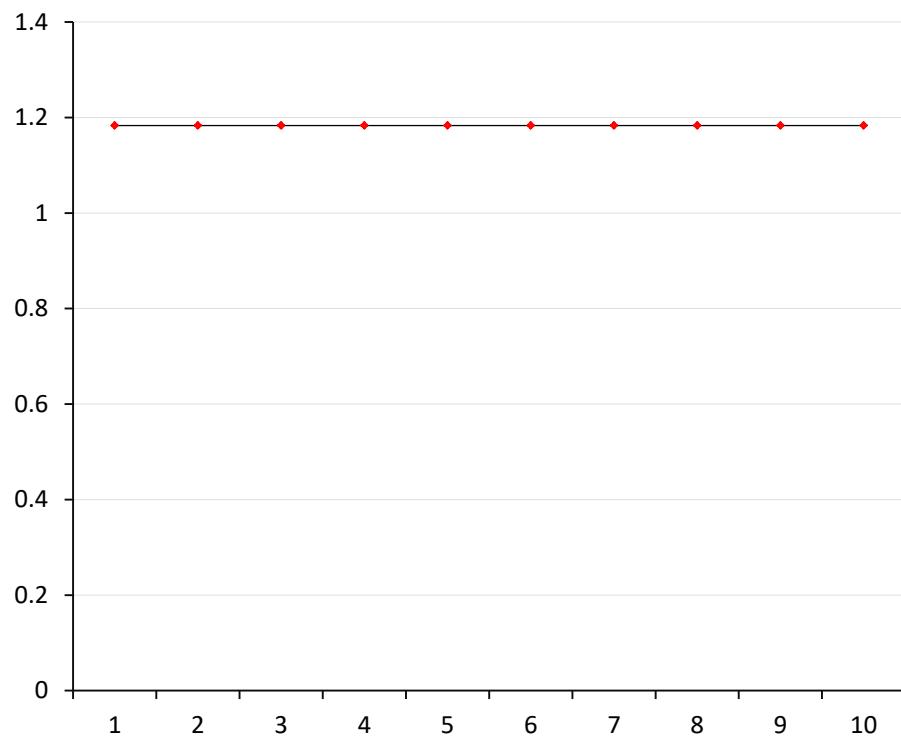
z	$y^z [x=7, b=6]$	$\log_6 y^z$	r
0	1	0	-
1	11	1.338290833	1.338290833
2	121	2.676581666	1.338290833
3	1331	4.014872499	1.338290833
4	15041	5.368206643	1.353334144
5	205451	6.827346556	1.459139913
6	2304401	8.176505598	1.349159042
7	25352411	9.514884495	1.338378897
8	323320521	10.935705964	1.420821469
9	4000530131	12.339654458	1.403948494
10	44010231441	13.678001092	1.338346634



$$r = (\log_6 y^z) - (\log_6 y^{z-1}), \text{ for } y = (7)_6$$

Septenary

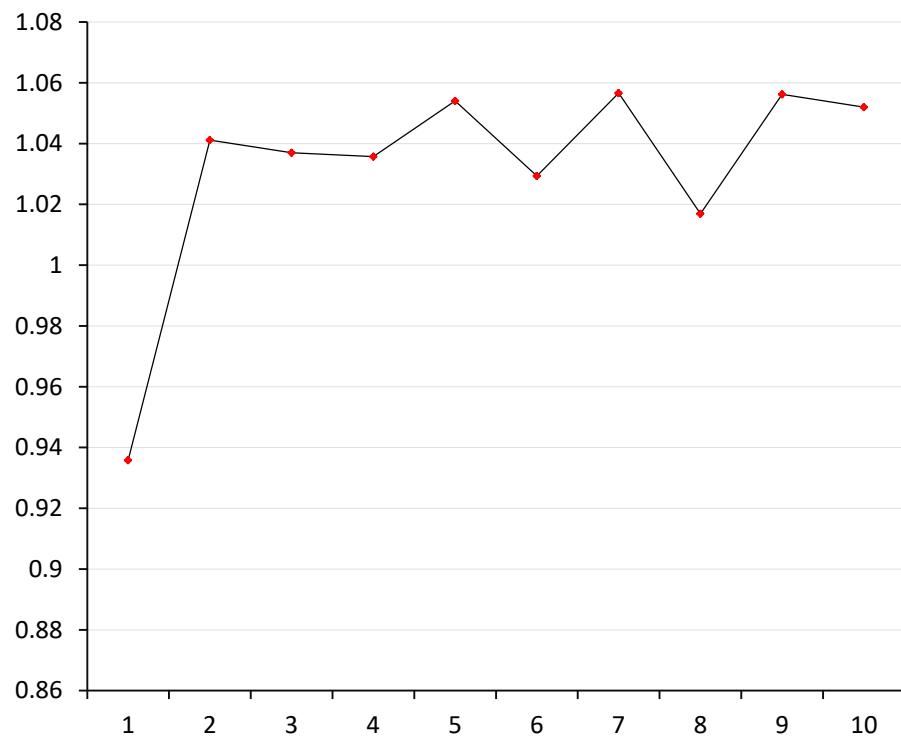
z	$y^z [x=7, b=7]$	$\log_7 y^z$	r
0	1	0	-
1	10	1.183294662	1.183294662
2	100	2.366589325	1.183294663
3	1000	3.549883987	1.183294662
4	10000	4.733178650	1.183294663
5	100000	5.916473312	1.183294662
6	1000000	7.099767975	1.183294663
7	10000000	8.283062637	1.183294662
8	100000000	9.466357300	1.183294663
9	1000000000	10.649651962	1.183294662
10	10000000000	11.832946625	1.183294663



$$r = (\log_7 y^z) - (\log_7 y^{z-1}), \text{ for } y = (7)_7$$

Octal

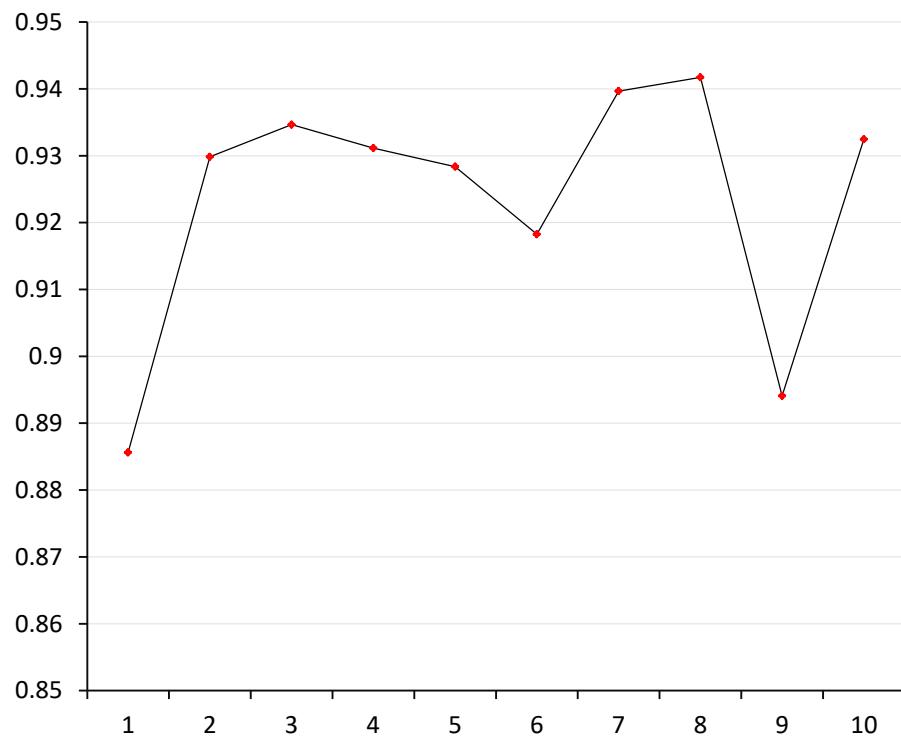
z	$y^z [x=7, b=8]$	$\log_8 y^z$	r
0	1	0	-
1	7	0.935784974	0.935784974
2	61	1.976912446	1.041127472
3	527	3.013886384	1.036973938
4	4541	4.049598107	1.035711723
5	40647	5.103620419	1.054022312
6	345621	6.132943784	1.029323365
7	3110367	7.189551129	1.056607345
8	25773301	8.206457997	1.016906868
9	231737507	9.262638775	1.056180778
10	2065435361	10.314599588	1.051960813



$$r = (\log_8 y^z) - (\log_8 y^{z-1}), \text{ for } y = (7)_8$$

Nonary

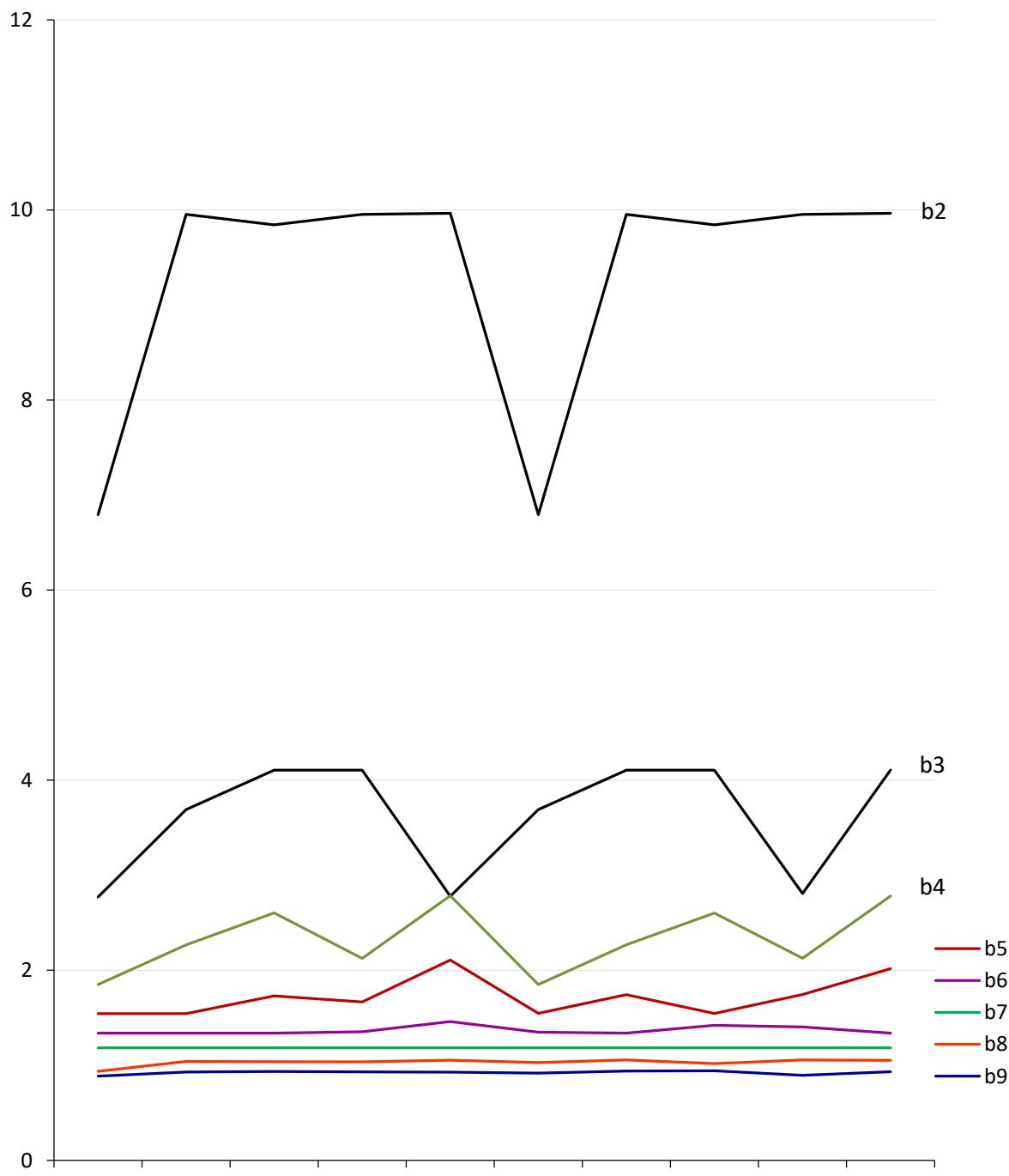
z	$y^z [x=7, b=9]$	$\log_9 y^z$	r
0	1	0	-
1	7	0.885621875	0.885621875
2	54	1.815464877	0.929843002
3	421	2.750120719	0.934655842
4	3257	3.681263121	0.931142402
5	25044	4.609628739	0.928365618
6	188341	5.527887114	0.918258375
7	1484617	6.467553445	0.939666331
8	11755734	7.409279801	0.941726356
9	83834661	8.303364747	0.894084946
10	650466167	9.235833266	0.932468519



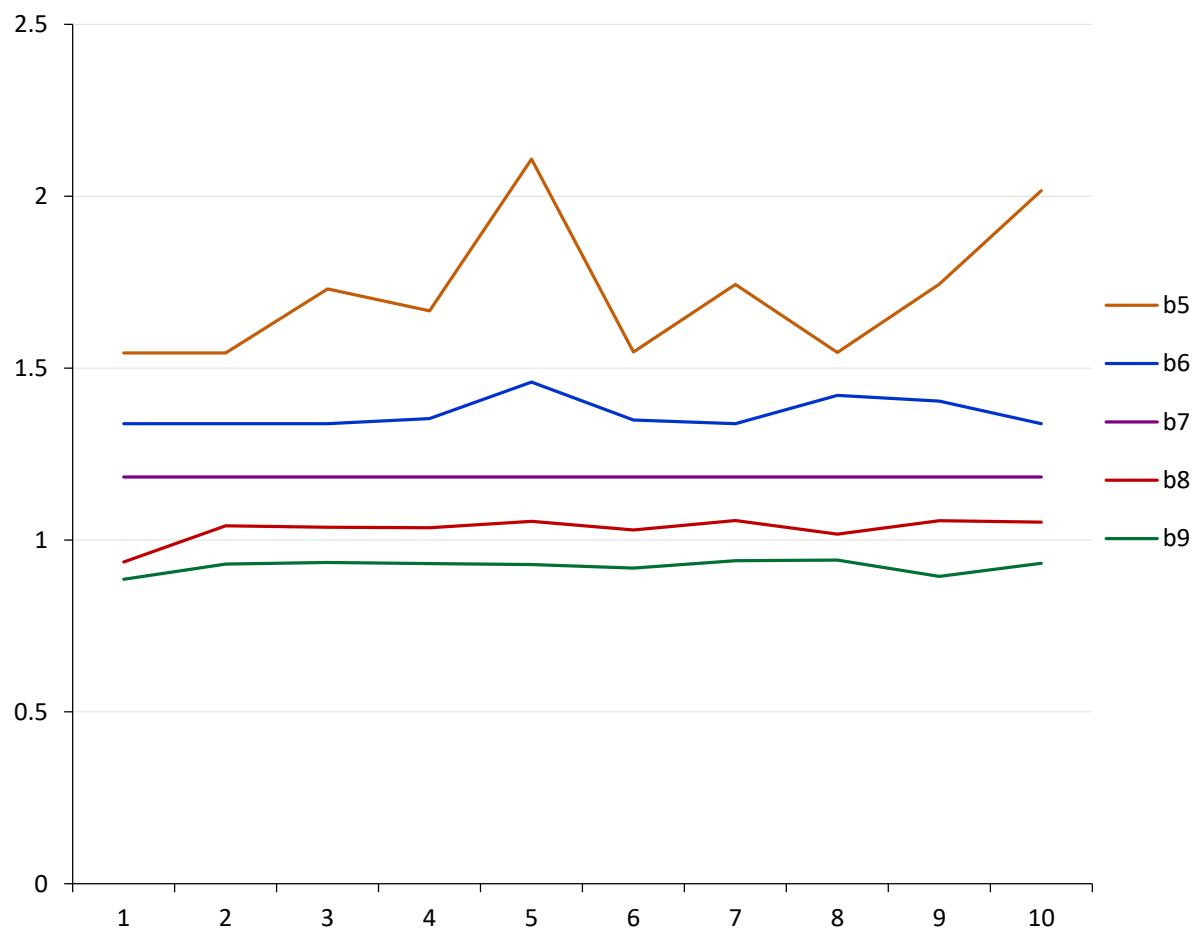
$$r = (\log_9 y^z) - (\log_9 y^{z-1}), \text{ for } y = (7)_9$$

Proportional graphs

The first graph below shows the distributions represented on pages 72-79 above with a proportional vertical axis for the full range $b=(2, \dots, 9)$. The second graph shows the relationships between the lower distributions for the range $b=(5, \dots, 9)$, with an expanded vertical scale (r):



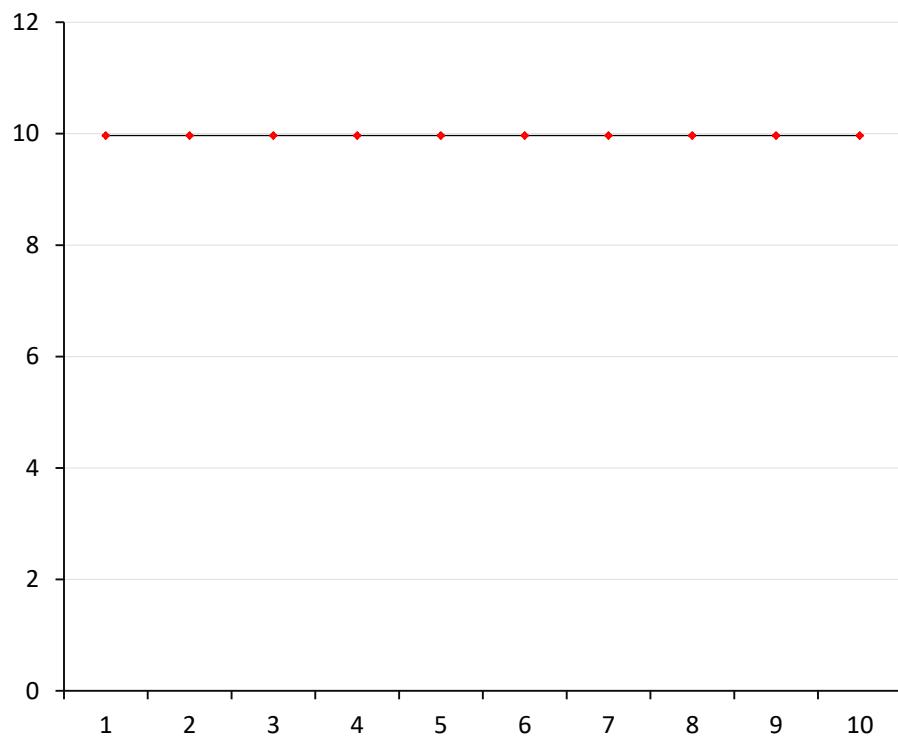
$$r = (\log_b y^z) - (\log_b y^{z-1}), \text{ for } y=(7)_b$$



$$\underline{x=8}$$

Binary

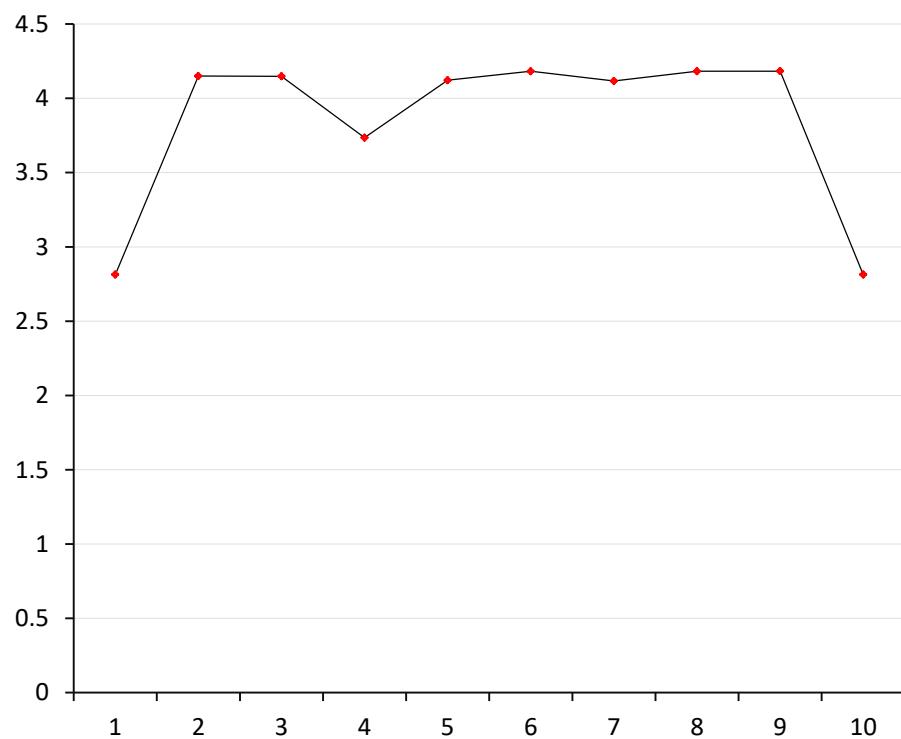
z	$y^z [x=8, b=2]$	$\log_2 y^z$	r
0	1	0	-
1	1000	9.965784285	9.965784285
2	1000000	19.931568569	9.965784284
3	10000000000	29.897352854	9.965784285
4	1000000000000000	39.863137139	9.965784285
5	10000000000000000000	49.828921423	9.965784284
6	10000000000000000000000000	59.794705708	9.965784285
7	10000000000000000000000000000000	69.760489993	9.965784285
8	100000000000000000000000000000000000	79.726274277	9.965784284
9	1000	89.692058562	9.965784285
10	1000	99.657842847	9.965784285



$$r = (\log_2 y^z) - (\log_2 y^{z-1}), \text{ for } y = (8)_2$$

Ternary

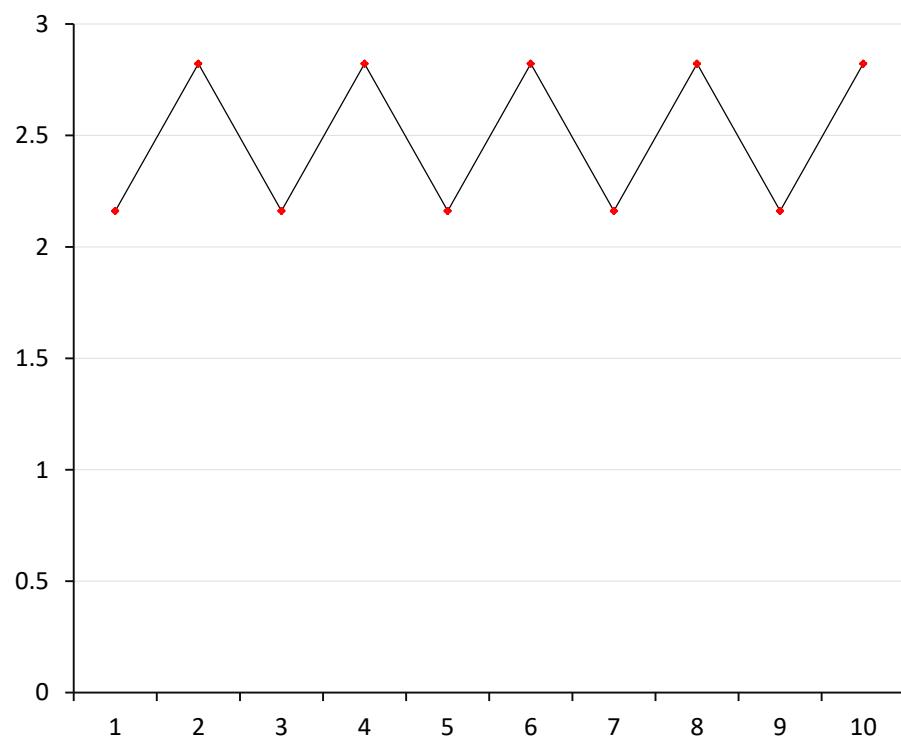
z	$y^z [x=8, b=3]$	$\log_3 y^z$	r
0	1	0	-
1	22	2.813588092	2.813588092
2	2101	6.963483642	4.149895550
3	200222	11.111455930	4.147972288
4	12121201	14.846426528	3.734970598
5	1122221122	18.968089032	4.121662504
6	111022121001	23.150109979	4.182020947
7	10221112202022	27.266649782	4.116539803
8	1011120101000101	31.448615202	4.181965420
9	100100112222002222	35.631266468	4.182651266
10	2202211102201212201	38.444858128	2.813591660



$$r = (\log_3 y^z) - (\log_3 y^{z-1}), \text{ for } y = (8)_3$$

Quaternary

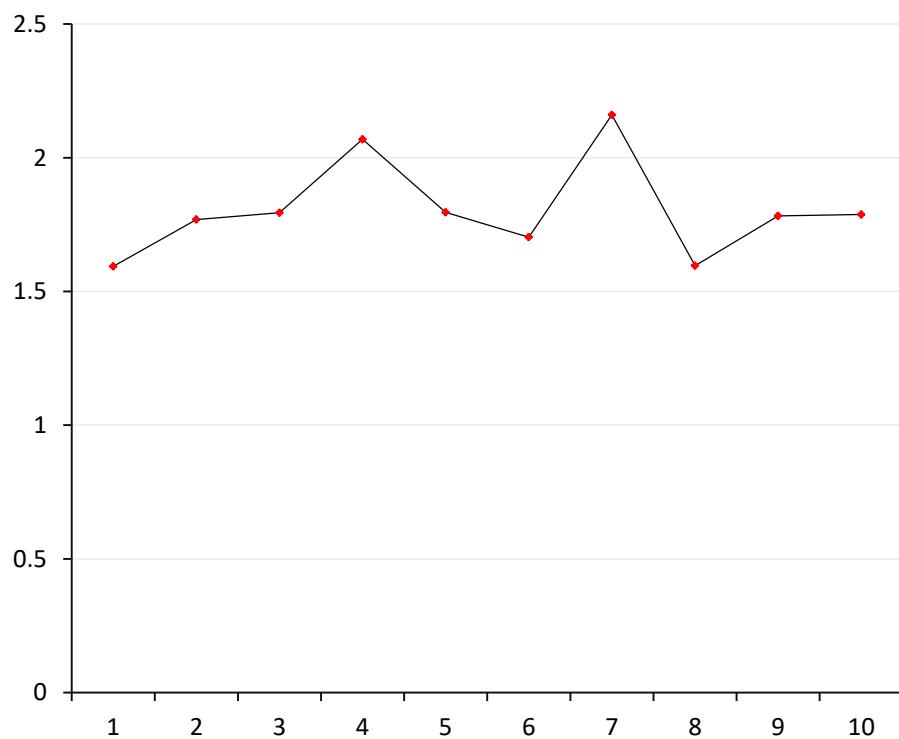
z	$y^z [x=8, b=4]$	$\log_4 y^z$	r
0	1	0	-
1	20	2.160964047	2.160964047
2	1000	4.982892142	2.821928095
3	20000	7.143856190	2.160964048
4	1000000	9.965784285	2.821928095
5	20000000	12.126748332	2.160964047
6	1000000000	14.948676427	2.821928095
7	20000000000	17.109640474	2.160964047
8	1000000000000	19.931568569	2.821928095
9	20000000000000	22.092532617	2.160964048
10	100000000000000	24.914460712	2.821928095



$$r = (\log_4 y^z) - (\log_4 y^{z-1}), \text{ for } y = (8)_4$$

Quinary

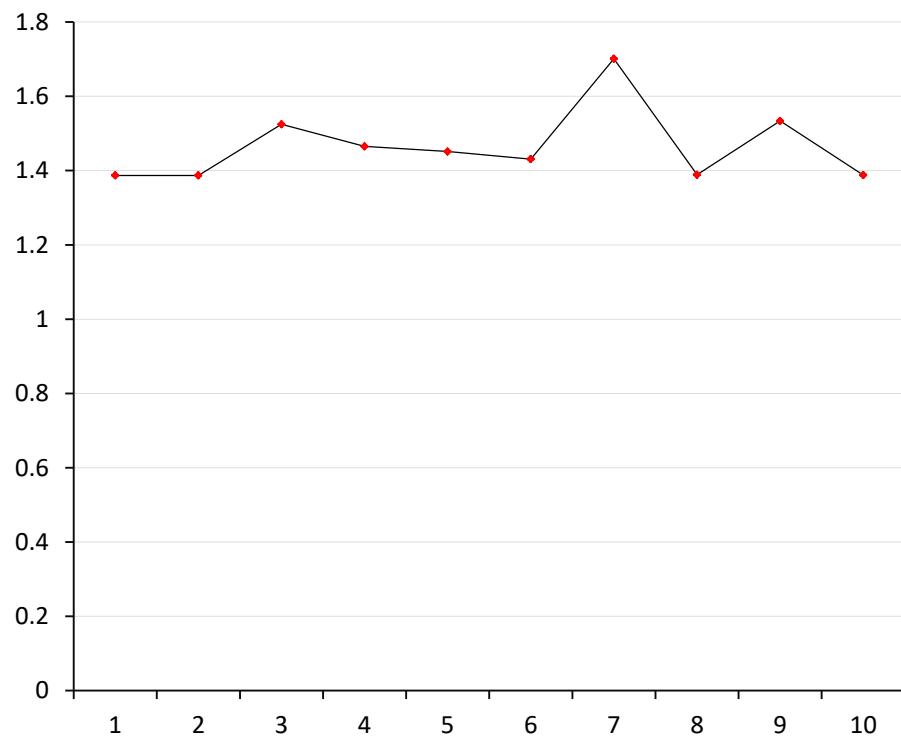
z	$y^z [x=8, b=5]$	$\log_5 y^z$	r
0	1	0	-
1	13	1.593692641	1.593692641
2	224	3.362444745	1.768752104
3	4022	5.156790769	1.794346024
4	112341	7.225686731	2.068895962
5	2022033	9.021543414	1.795856683
6	31342034	10.724533429	1.702990015
7	1014102102	12.884789944	2.160256515
8	13243332331	14.481304226	1.596514282
9	233324431403	16.263874194	1.782569968
10	4144334214244	18.051496764	1.787622570



$$r = (\log_5 y^z) - (\log_5 y^{z-1}), \text{ for } y = (8)_5$$

Senary

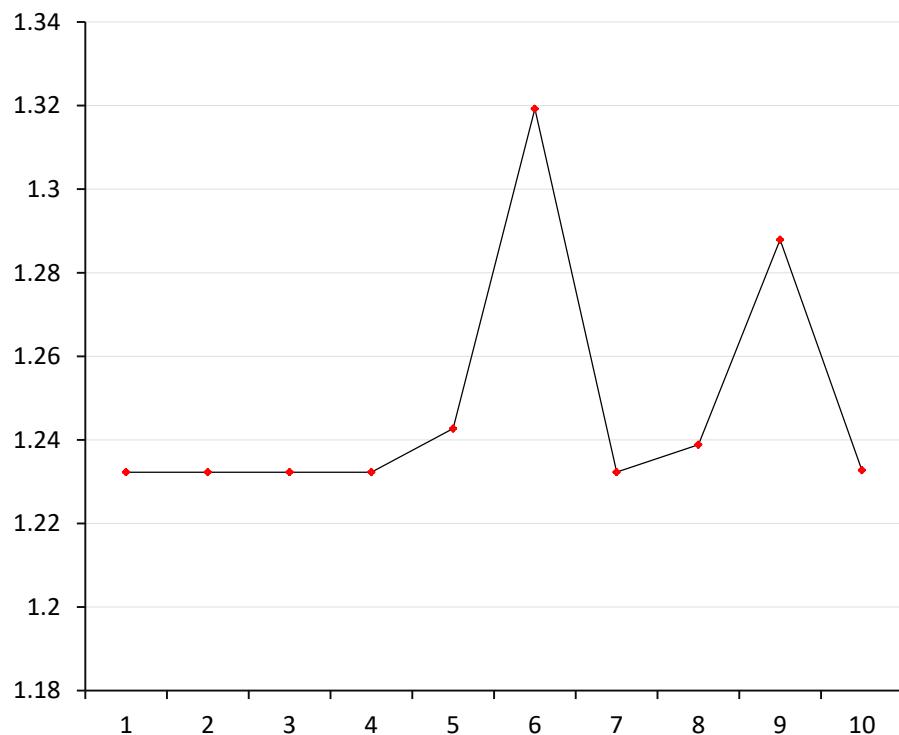
z	$y^z [x=8, b=6]$	$\log_6 y^z$	r
0	1	0	-
1	12	1.386852807	1.386852807
2	144	2.773705614	1.386852807
3	2212	4.298374026	1.524668412
4	30544	5.763565771	1.465191745
5	411412	7.214891645	1.451325874
6	5341344	8.645684943	1.430793298
7	112541012	10.346717058	1.701032115
8	1355332144	11.735566490	1.388849432
9	21152430212	13.269091665	1.533525175
10	254314002544	14.657009505	1.387917840



$$r = (\log_6 y^z) - (\log_6 y^{z-1}), \text{ for } y = (8)_6$$

Septenary

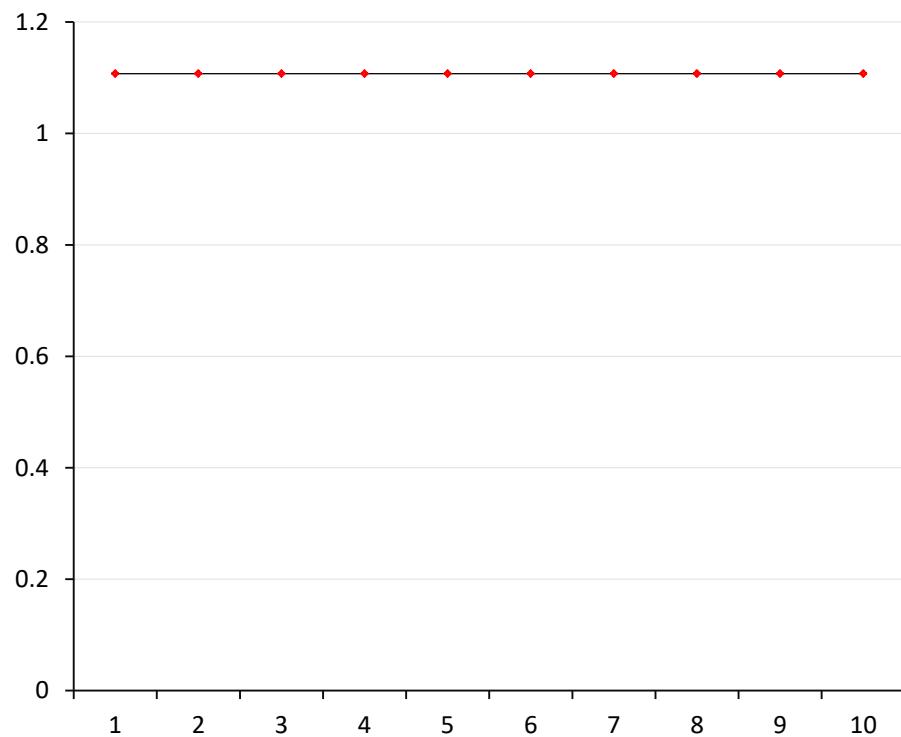
z	$y^z [x=8, b=7]$	$\log_7 y^z$	r
0	1	0	-
1	11	1.232274406	1.232274406
2	121	2.464548812	1.232274406
3	1331	3.696823218	1.232274406
4	14641	4.929097623	1.232274405
5	164351	6.171795583	1.242697960
6	2141161	7.491023555	1.319227972
7	23553101	8.723305161	1.232281606
8	262414111	9.962142599	1.238837438
9	3216555221	11.250045038	1.287902439
10	35415440431	12.482803353	1.232758315



$$r = (\log_7 y^z) - (\log_7 y^{z-1}), \text{ for } y = (8)_7$$

Octal

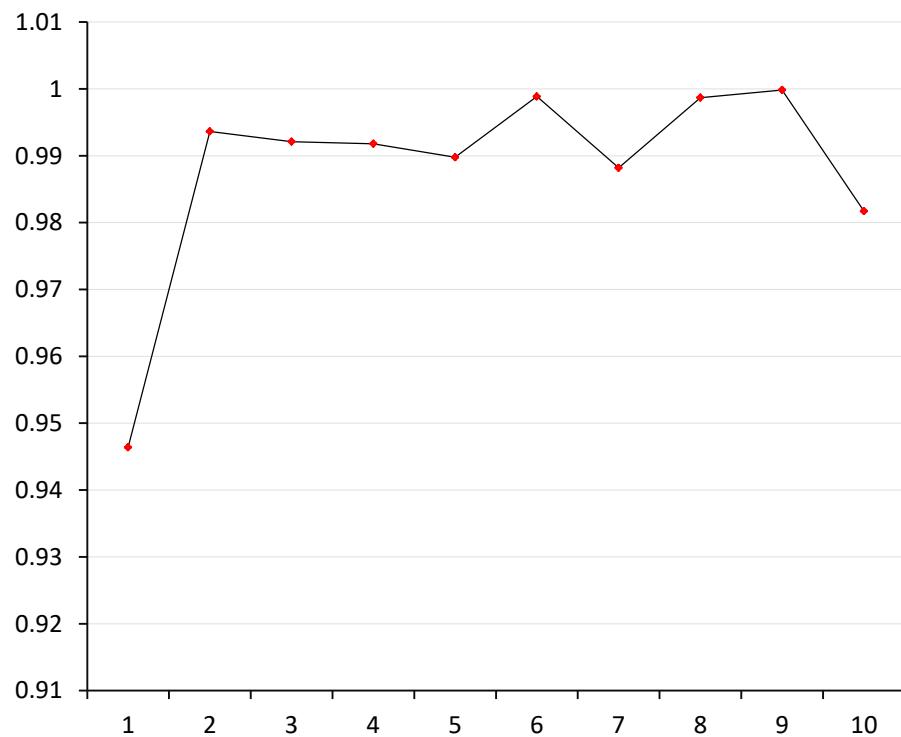
z	$y^z [x=8, b=8]$	$\log_8 y^z$	r
0	1	0	-
1	10	1.107309365	1.107309365
2	100	2.214618730	1.107309365
3	1000	3.321928095	1.107309365
4	10000	4.429237460	1.107309365
5	100000	5.536546825	1.107309365
6	1000000	6.643856190	1.107309365
7	10000000	7.751165555	1.107309365
8	100000000	8.858474920	1.107309365
9	1000000000	9.965784285	1.107309365
10	10000000000	11.073093650	1.107309365



$$r = (\log_8 y^z) - (\log_8 y^{z-1}), \text{ for } y = (8)_8$$

Nonary

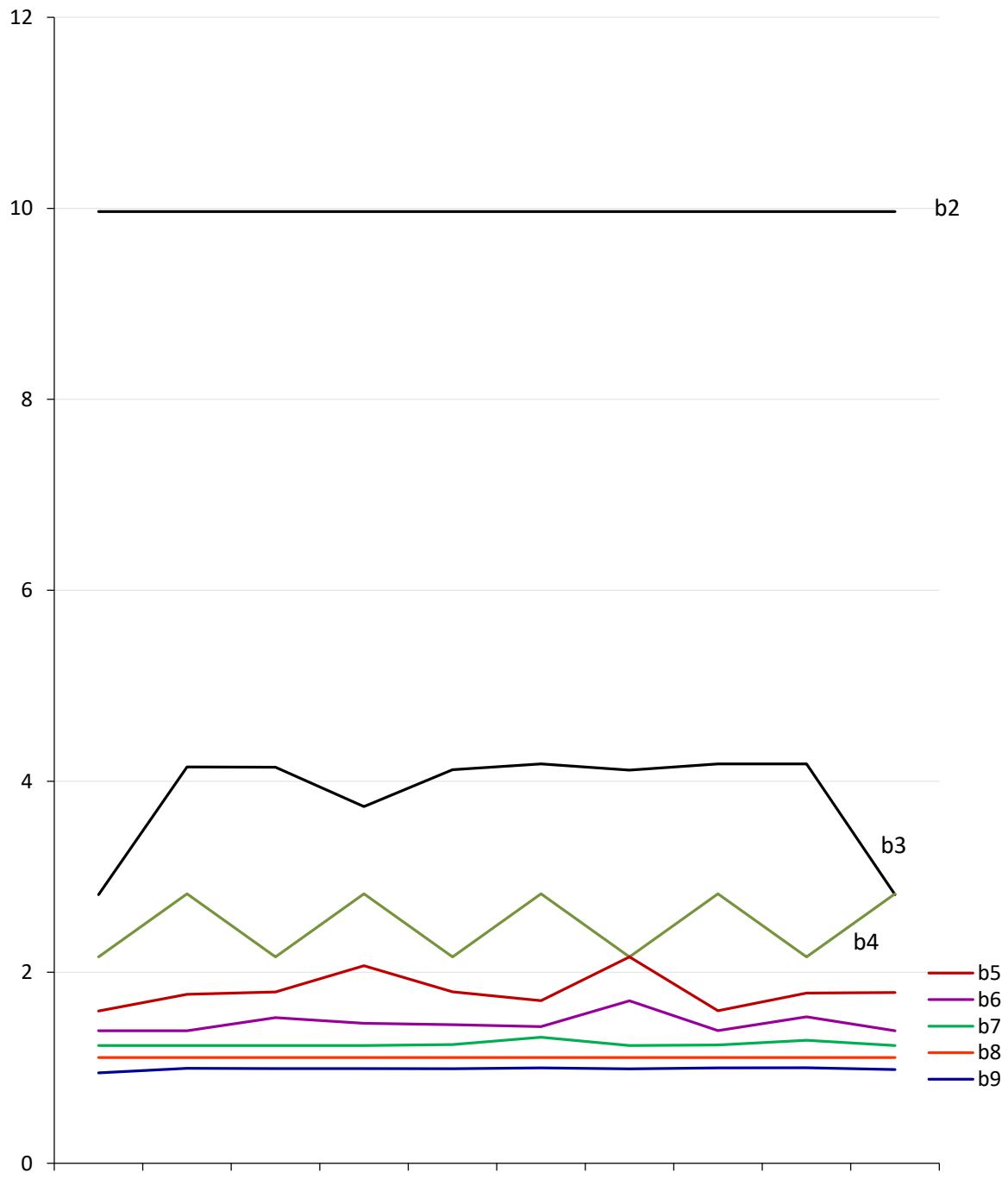
z	$y^z [x=8, b=9]$	$\log_9 y^z$	r
0	1	0	-
1	8	0.946394630	0.946394630
2	71	1.940029217	0.993634587
3	628	2.932126389	0.992097172
4	5551	3.923919958	0.991793569
5	48848	4.913684667	0.989764709
6	438531	5.912543450	0.998858783
7	3845668	6.900731966	0.988188516
8	34511011	7.899415094	0.998683128
9	310488088	8.899252401	0.999837307
10	2684381781	9.880972802	0.981720401



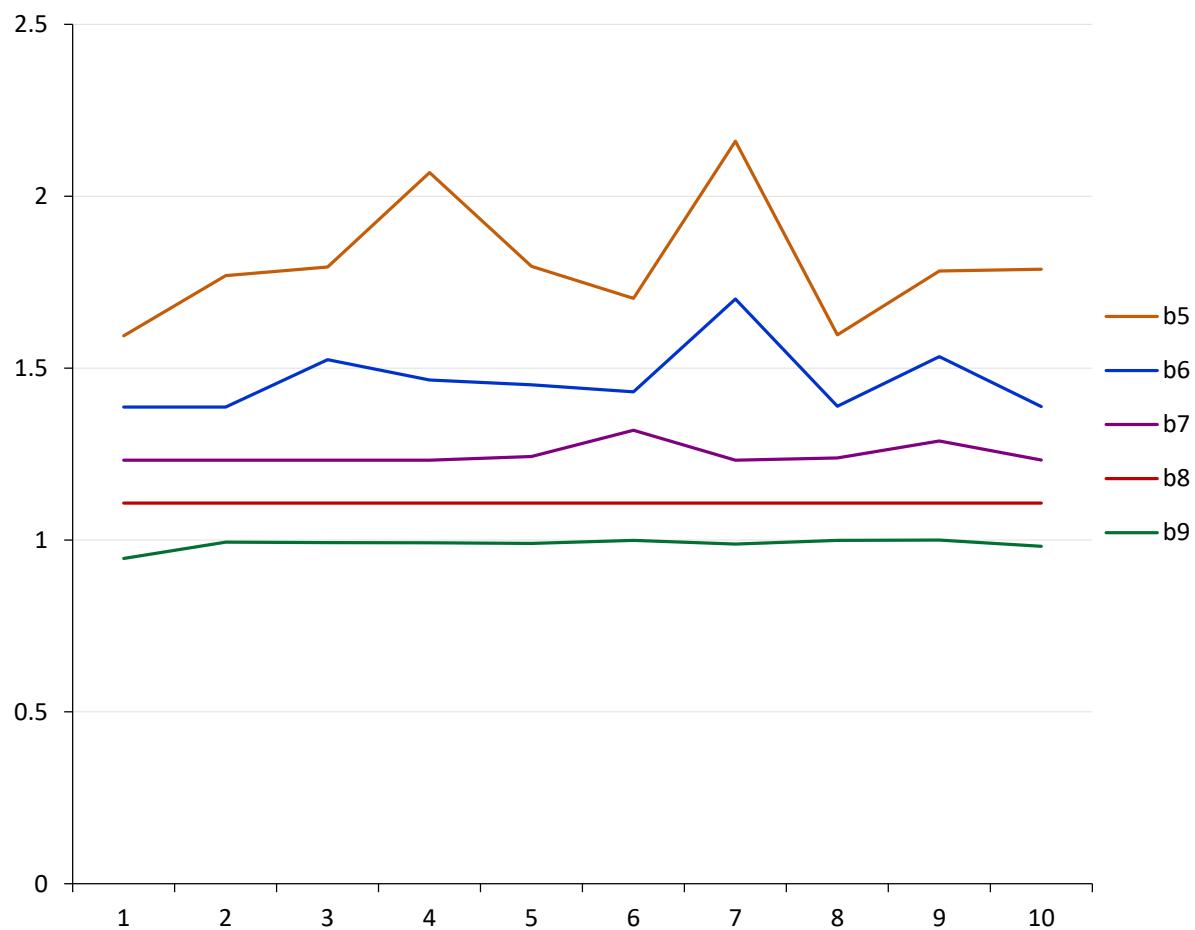
$$r = (\log_9 y^z) - (\log_9 y^{z-1}), \text{ for } y = (8)_9$$

Proportional graphs

The first graph below shows the distributions represented on pages 82-89 above with a proportional vertical axis for the full range $b=(2, \dots, 9)$. The second graph shows the relationships between the lower distributions for the range $b=(5, \dots, 9)$, with an expanded vertical scale (r):



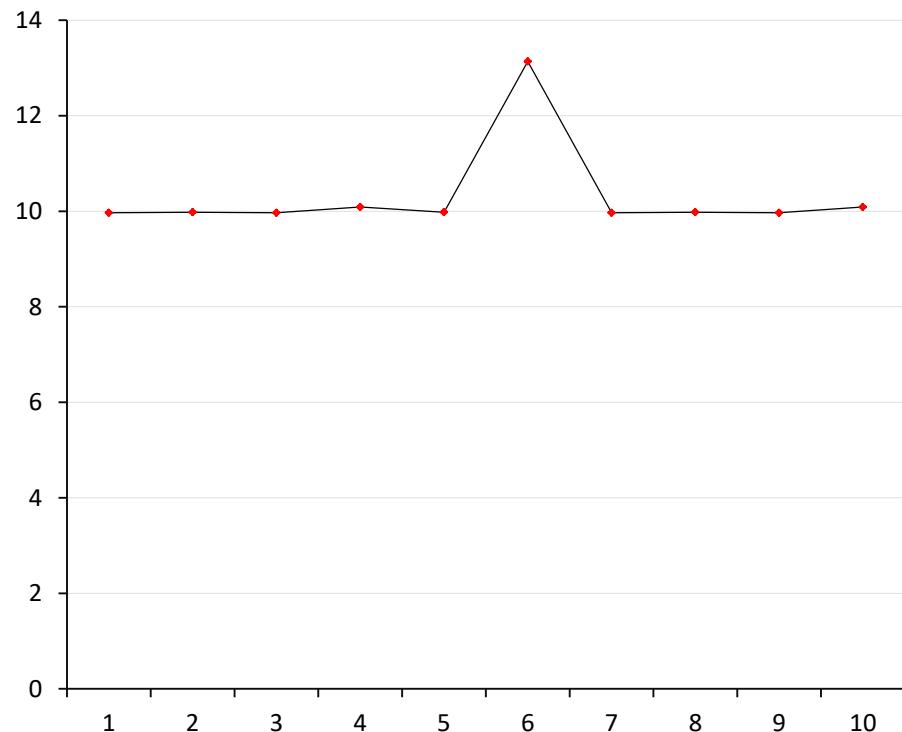
$$r = (\log_b y^z) - (\log_b y^{z-1}), \text{ for } y=(8)_b$$



$$\underline{x=9}$$

Binary

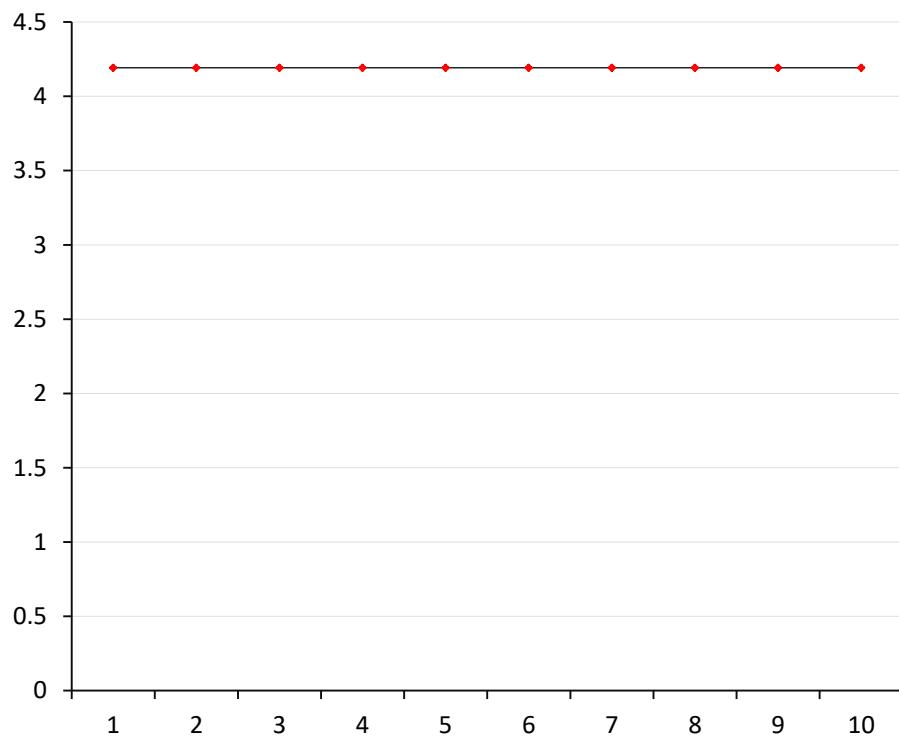
z	$y^z [x=9, b=2]$	$\log_2 y^z$	r
0	1	0	-
1	1001	9.967226259	9.967226259
2	1010001	19.945925291	9.978699032
3	1011011001	29.913151550	9.967226259
4	1100110100001	40.000785056	10.087633506
5	1110011010101001	49.979495410	9.978710354
6	1000000110111110001	63.116633962	13.137138552
7	10010001111101101111001	73.083860222	9.967226260
8	10100100001101011101000001	83.062571949	9.978711727
9	10111000101111001000101001001	93.029912361	9.967340412
10	1100111110101000001101110010001	103.117420185	10.087507824



$$r = (\log_2 y^z) - (\log_2 y^{z-1}), \text{ for } y = (9)_2$$

Ternary

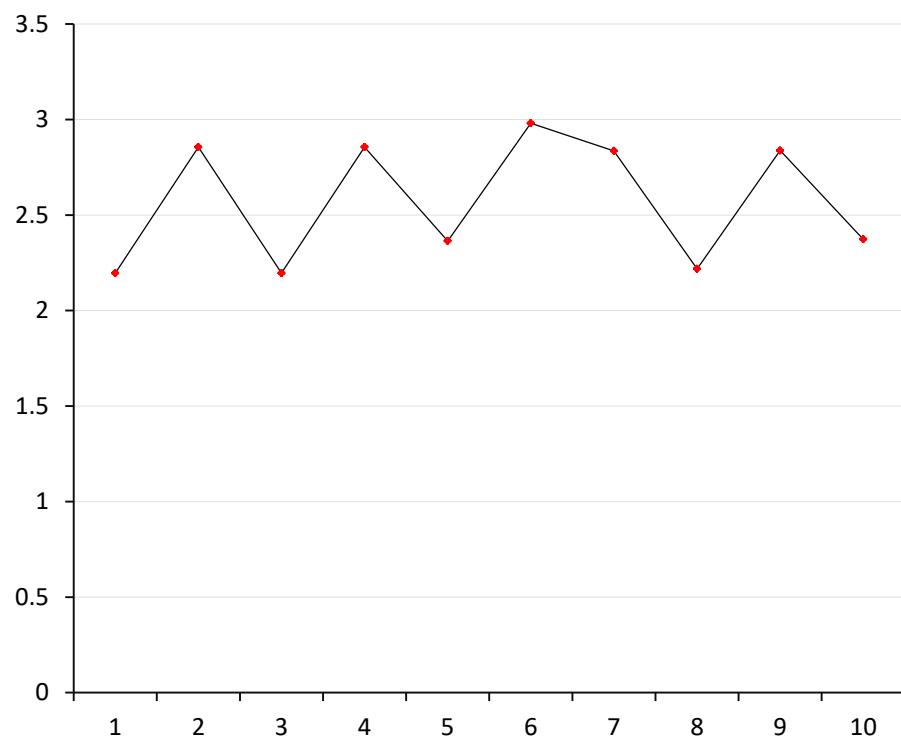
z	$y^z [x=9, b=3]$	$\log_3 y^z$	r
0	1	0	-
1	100	4.191806549	4.191806549
2	10000	8.383613097	4.191806548
3	1000000	12.575419646	4.191806549
4	1000000000	16.767226194	4.191806548
5	100000000000	20.959032743	4.191806549
6	10000000000000	25.150839291	4.191806548
7	1000000000000000	29.342645840	4.191806549
8	10000000000000000	33.534452389	4.191806549
9	100000000000000000	37.726258937	4.191806548
10	1000000000000000000	41.918065486	4.191806549



$$r = (\log_3 y^z) - (\log_3 y^{z-1}), \text{ for } y = (9)_3$$

Quaternary

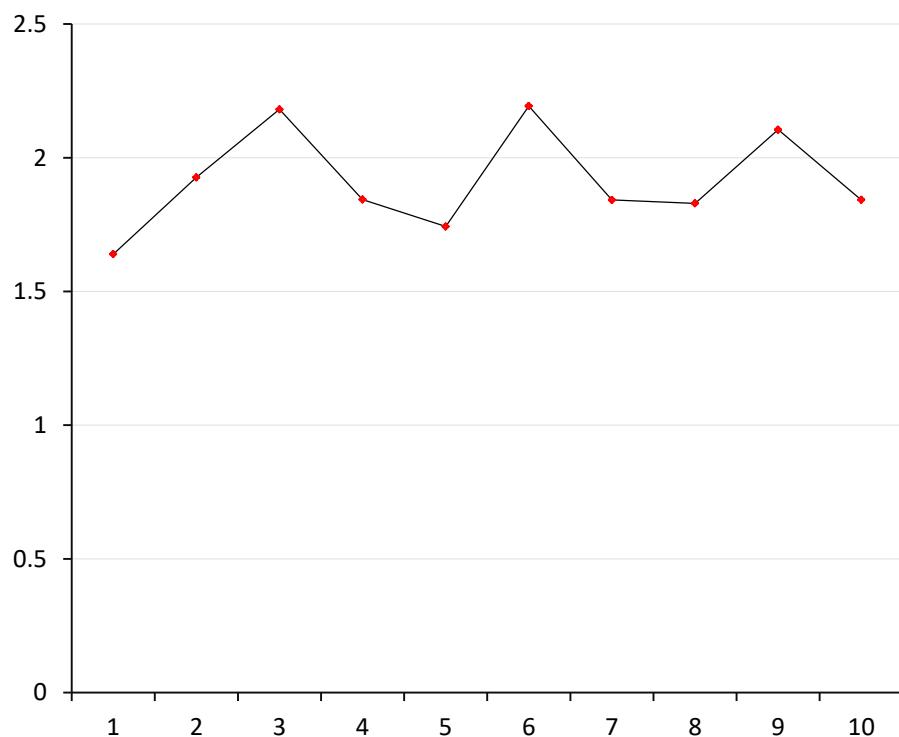
z	$y^z [x=9, b=4]$	$\log_4 y^z$	r
0	1	0	-
1	21	2.196158711	2.196158711
2	1101	5.052299377	2.856140666
3	23121	7.248458088	2.196158711
4	1212201	10.104598754	2.856140666
5	32122221	12.468534156	2.363935402
6	2001233301	15.449121109	2.980586953
7	102033231321	18.285124073	2.836002964
8	2210031131001	20.503601915	2.218477842
9	113011321011021	23.341730316	2.838128401
10	3033311001232101	25.714907423	2.373177107



$$r = (\log_4 y^z) - (\log_4 y^{z-1}), \text{ for } y = (9)_4$$

Quinary

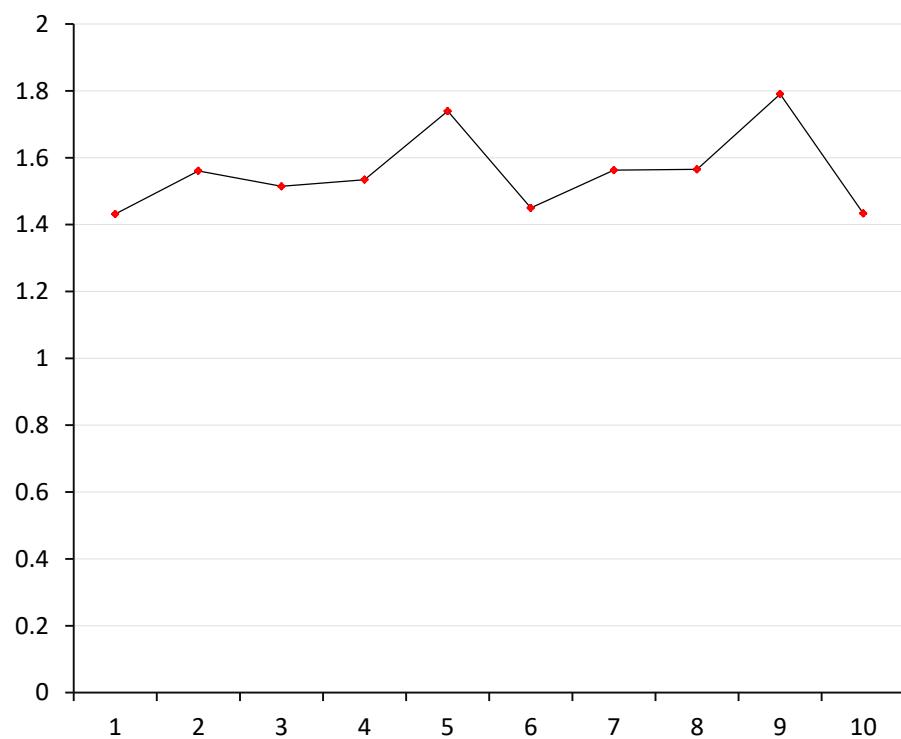
z	$y^z [x=9, b=5]$	$\log_5 y^z$	r
0	1	0	-
1	14	1.639738513	1.639738513
2	311	3.566333853	1.926595340
3	10404	5.747314361	2.180980508
4	202221	7.590921242	1.843606881
5	3342144	9.333769858	1.742848616
6	114001231	11.526831611	2.193061753
7	2211023334	13.369090616	1.842259005
8	42004443341	15.198499460	1.829408844
9	1243134423424	17.303343516	2.104844056
10	24120104100101	19.145856181	1.842512665



$$r = (\log_5 y^z) - (\log_5 y^{z-1}), \text{ for } y = (9)_5$$

Senary

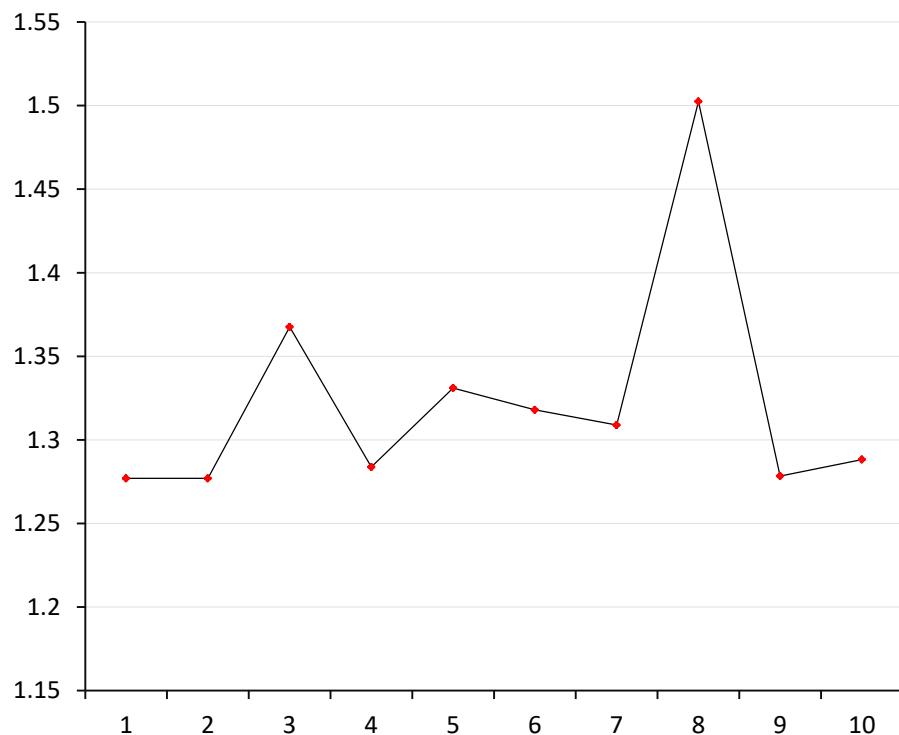
z	$y^z [x=9, b=6]$	$\log_6 y^z$	r
0	1	0	-
1	13	1.431525493	1.431525493
2	213	2.992194130	1.560668637
3	3213	4.506721185	1.514527055
4	50213	6.041005739	1.534284554
5	1133213	7.780378872	1.739373133
6	15220213	9.230108834	1.449729962
7	250303213	10.792845761	1.562736927
8	4134350213	12.358018133	1.565172372
9	102235433213	14.148408084	1.790389951
10	1333553520213	15.581817088	1.433409004



$$r = (\log_6 y^z) - (\log_6 y^{z-1}), \text{ for } y = (9)_6$$

Septenary

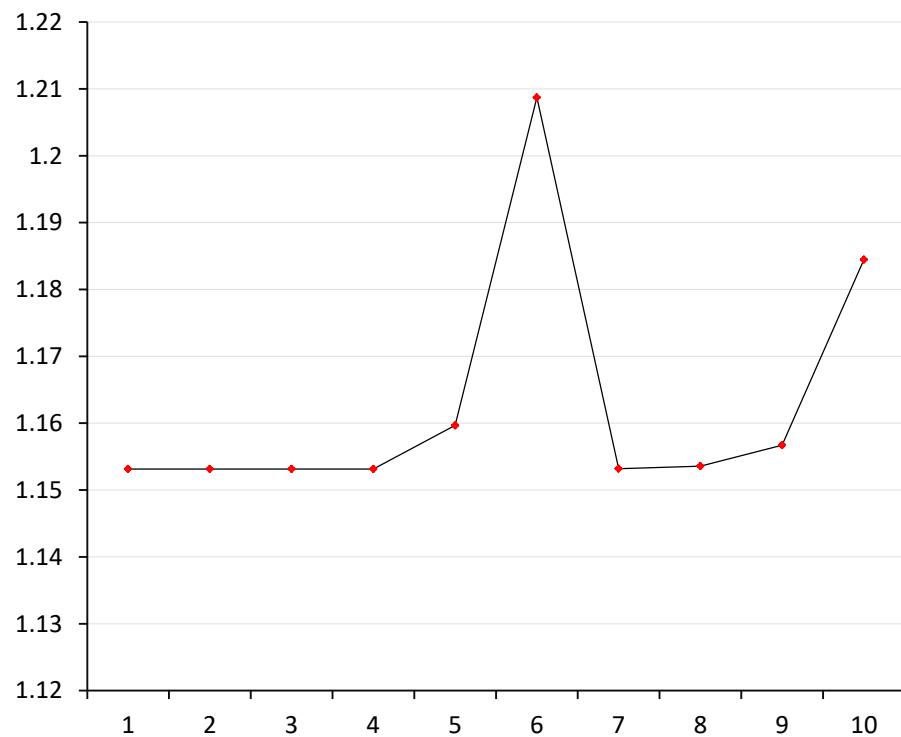
z	$y^z [x=9, b=7]$	$\log_7 y^z$	r
0	1	0	-
1	12	1.276989408	1.276989408
2	144	2.553978817	1.276989409
3	2061	3.921530799	1.367551982
4	25062	5.205331828	1.283801029
5	334104	6.536379702	1.331047874
6	4342251	7.854372640	1.317992938
7	55440342	9.163227855	1.308855215
8	1031614434	10.665647041	1.502419186
9	12413006541	11.944030759	1.278383718
10	152256115122	13.232281008	1.288250249



$$r = (\log_7 y^z) - (\log_7 y^{z-1}), \text{ for } y = (9)_7$$

Octal

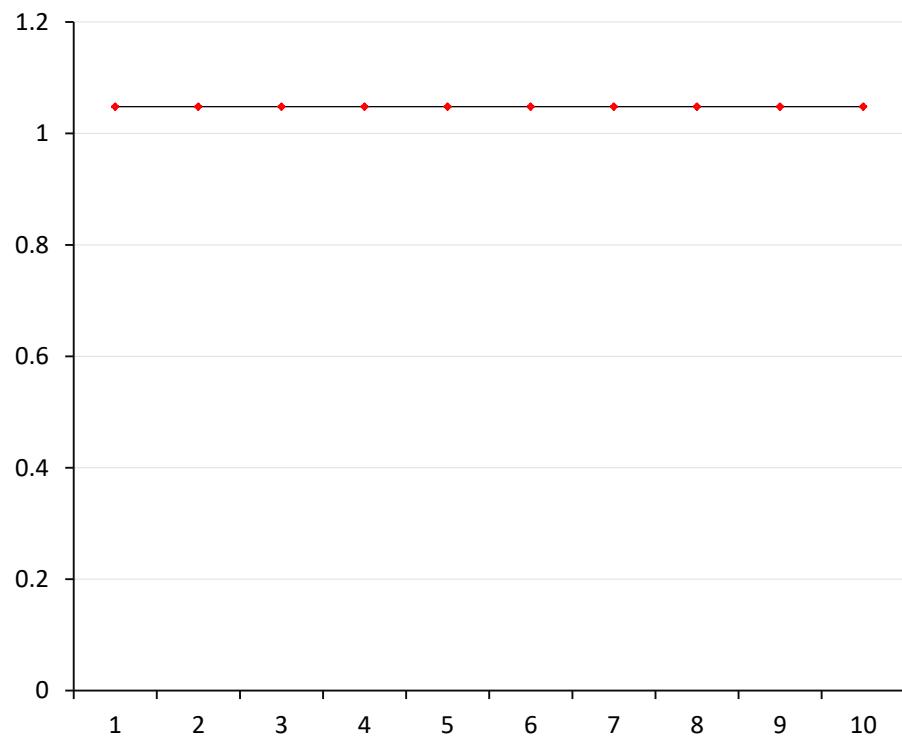
z	$y^z [x=9, b=8]$	$\log_8 y^z$	r
0	1	0	-
1	11	1.153143873	1.153143873
2	121	2.306287746	1.153143873
3	1331	3.459431619	1.153143873
4	14641	4.612575492	1.153143873
5	163251	5.772244101	1.159668609
6	2015761	6.980964388	1.208720287
7	22175571	8.134155972	1.153191584
8	244153501	9.287737742	1.153581770
9	2705710511	10.444453448	1.156715706
10	31765015621	11.628906559	1.184453111



$$r = (\log_8 y^z) - (\log_8 y^{z-1}), \text{ for } y = (9)_8$$

Nonary

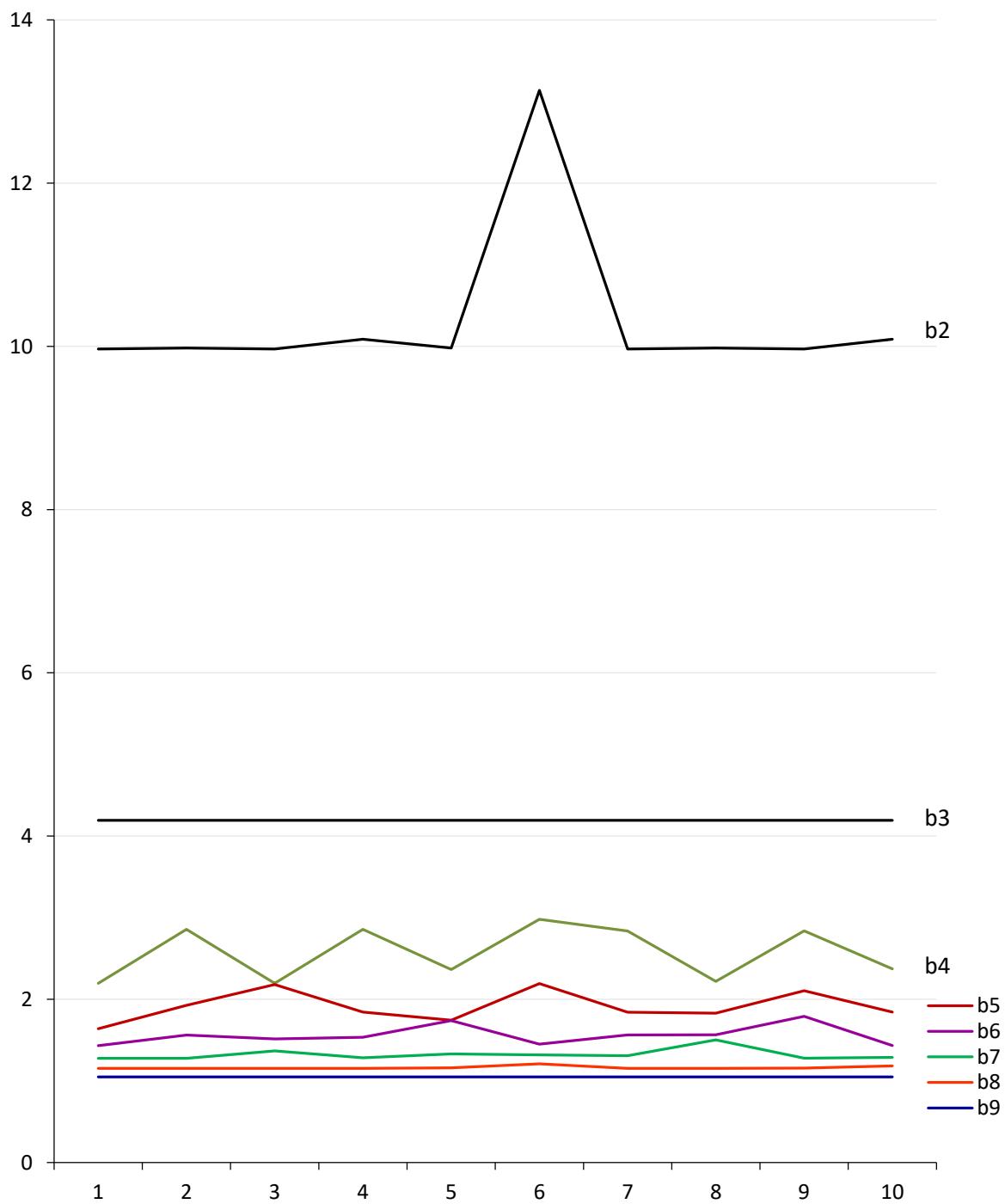
z	$y^z [x=9, b=9]$	$\log_9 y^z$	r
0	1	0	-
1	10	1.047951637	1.047951637
2	100	2.095903274	1.047951637
3	1000	3.143854911	1.047951637
4	10000	4.191806549	1.047951638
5	100000	5.239758186	1.047951637
6	1000000	6.287709823	1.047951637
7	10000000	7.335661460	1.047951637
8	100000000	8.383613097	1.047951637
9	1000000000	9.431564734	1.047951637
10	10000000000	10.479516371	1.047951637



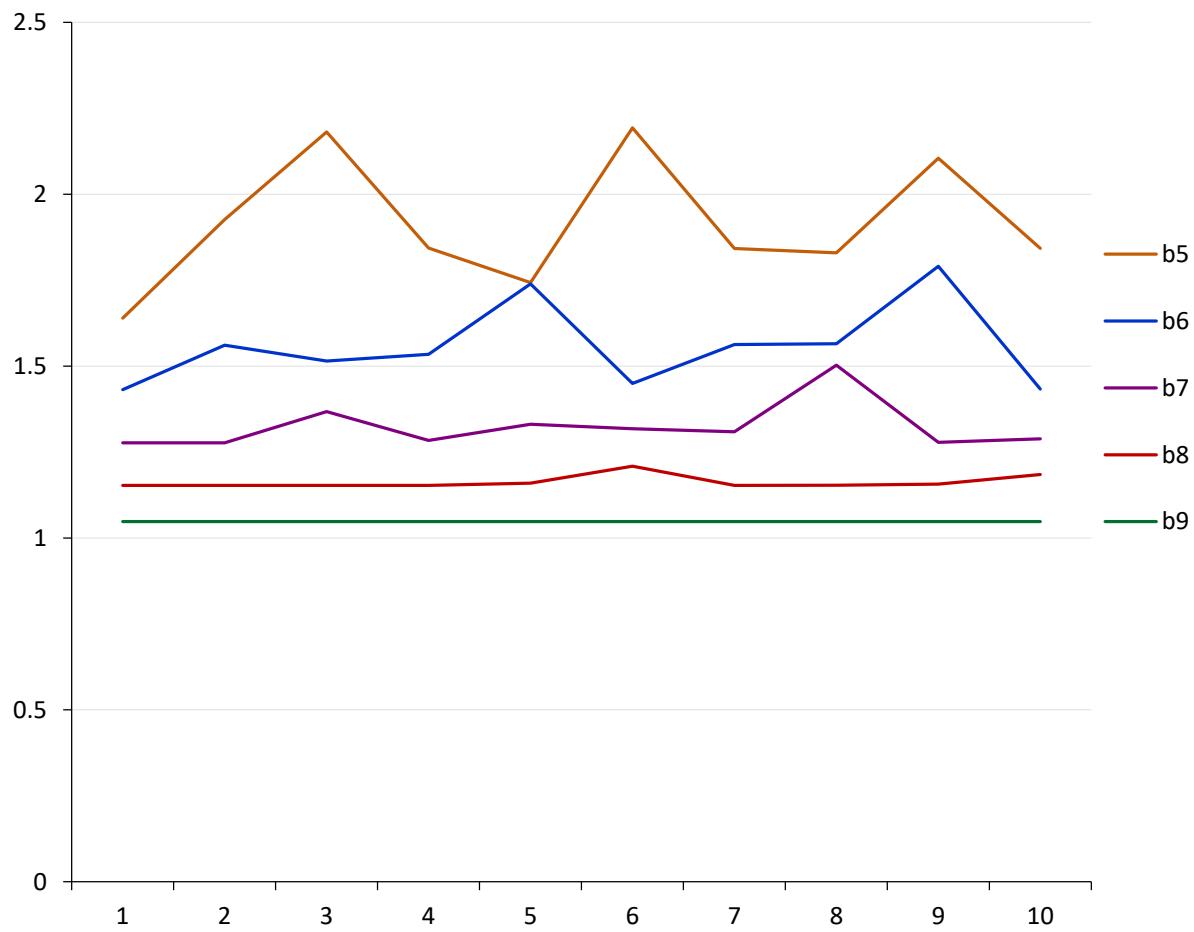
$$r = (\log_9 y^z) - (\log_9 y^{z-1}), \text{ for } y = (9)_9$$

Proportional graphs

The first graph below shows the distributions represented on pages 92-99 above with a proportional vertical axis for the full range $b=(2, \dots, 9)$. The second graph shows the relationships between the lower distributions for the range $b=(5, \dots, 9)$, with an expanded vertical scale (r):



$$r = (\log_b y^z) - (\log_b y^{z-1}), \text{ for } y=(9)_b$$



Variation factors across the series

The following tables repeat the analyses made on p.15 above of the ranges in variation across the $x=10$ series, this time for the extended series $x=(2, [...], 9)$. That is, the ratios of the maximum against the minimum (or ‘baseline’) variation factors (r_{max}/r_u) are represented for each radical derivation of x in its exponential series, as described by y^z . In each table the ratio r_{max}/r_u is ascribed the value q . For each instance of q the value of z at which r_{max} occurs in the series is added in parentheses. In some cases the same value r_{max} occurs at several values of z . Where other values of z produce values of r very close to r_{max} , these are italicised. Where $z=n$ here, the graph is a horizontal straight line and $r_{max}=r_u$ – the same for each value of z . These tables do not repeat the calculations of r_v (at $z=7$) found on page 15 above, as the statistical frequency of r_v is not repeated across all the series. For completeness the final table shows values of q for the $x=10$ series.

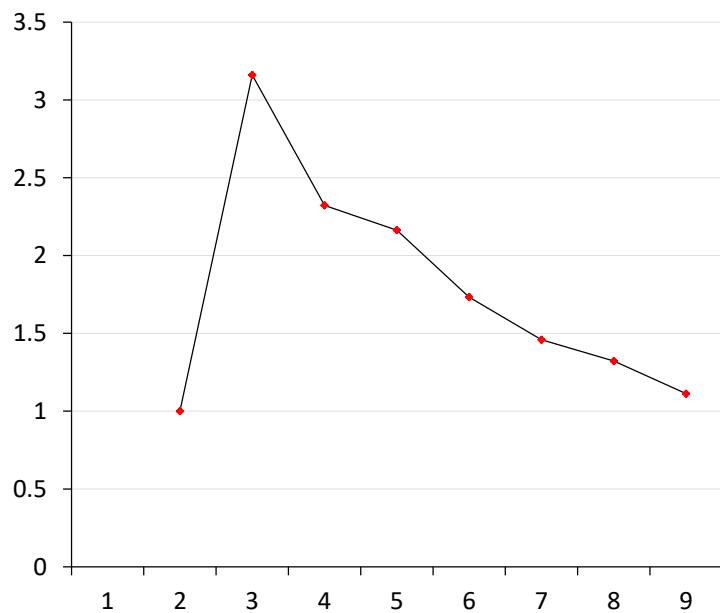
As a reminder, in each case r is given by: $r = (\log_b y^z) - (\log_b y^{z-1})$, for $y=(x_{10})_b$

Graphs of the distributions are shown below the tables, with q as the vertical and b as the horizontal axis.* Each *point* on each of these graphs therefore indicates the value of the *greatest difference* between r and its baseline value at $z=1$ occurring in each of the complete distributions represented on pages 5-99 above. Where $q=1$, it indicates a straight line distribution in the original graph. The only graph which does not exhibit at least one instance of $q=1$, is that for $x=10$, as this would occur at $b=10$, which is beyond our range. Had we included base-10, the value of q in each of the graphs at $b=10$ would be 1.

* The value ‘1’ on the horizontal b -axis has no meaning on these graphs and should be ignored.

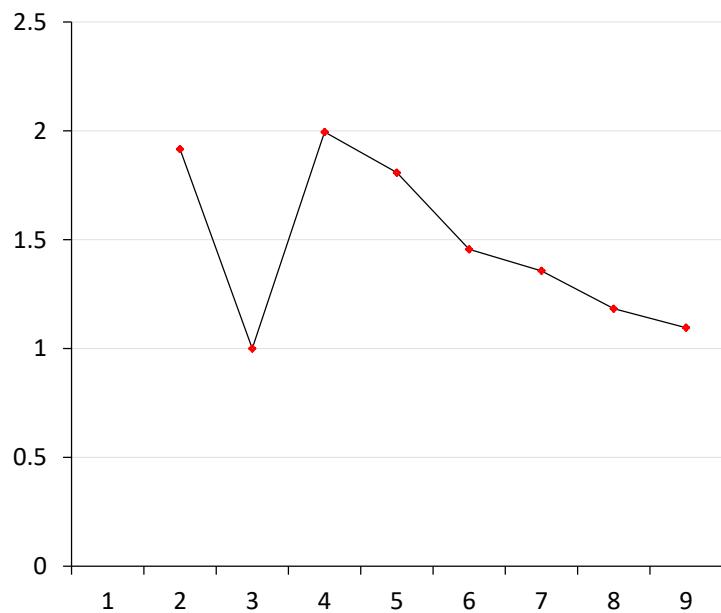
$x=2$

b	$r_u(z=1)$	$r_{max}(z=n)$	$q = r_{max}/r_u$
2	3.321928095	3.321928095	1.000 ($z=n$)
3	0.630929754	1.993594337	3.160 ($z=8$)
4	0.5	1.160964048	2.322 ($z=2,4,6,8,10$)
5	0.430676558	0.931446144	2.163 ($z=7$)
6	0.386852807	0.670603548	1.733 ($z=8$)
7	0.356207187	0.519860032	1.459 ($z=3,6$)
8	0.333333333	0.440642698	1.322 ($z=3,6,9$)
9	0.315464877	0.350665791	1.112 ($z=10$)



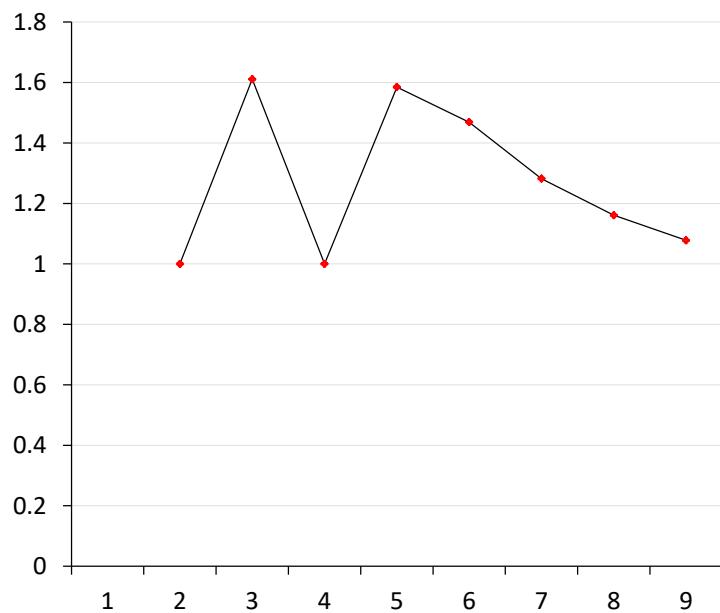
$x=3$

b	$r_u (z=1)$	$r_{max} (z=n)$	$q = r_{max}/r_u$
2	3.459431619	6.628201771	1.916 ($z=7$)
3	2.095903274	2.095903275	1.000 ($z=n$)
4	0.792481250	1.581042124	1.995 ($z=4$)
5	0.682606194	1.233918667	1.808 ($z=3,6$)
6	0.613147193	0.893028377	1.456 ($z=4$)
7	0.564575034	0.765909573	1.357 ($z=6,9$)
8	0.528320834	0.624823040	1.183 ($z=2,4,6$)
9	0.5	0.547951637	1.096 ($z=2,4,6,8,10$)



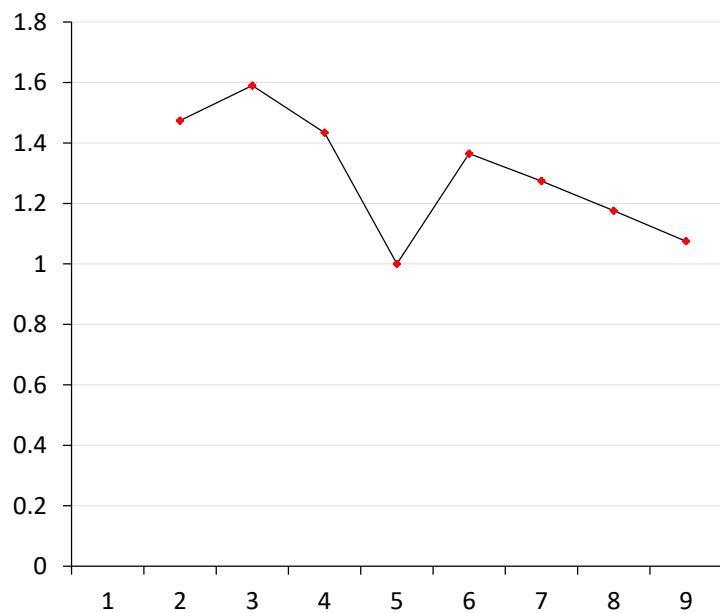
$\chi = 4$

b	$r_u (z=1)$	$r_{max} (z=n)$	$q = r_{max}/r_u$
2	6.643856190	6.643856190	1.000 ($z=n$)
3	2.182658339	3.517042535	1.611 ($z=4,8$)
4	1.660964047	1.660964048	1.000 ($z=n$)
5	0.861353116	1.365178588	1.585 ($z=4,7$)
6	0.773705614	1.136805449	1.469 ($z=4$)
7	0.712414374	0.913387948	1.282 ($z=9$)
8	0.666666667	0.773976032	1.161 ($z=2,3,5,6,8,9$)
9	0.630929754	0.679952987	1.078 ($z=8$)



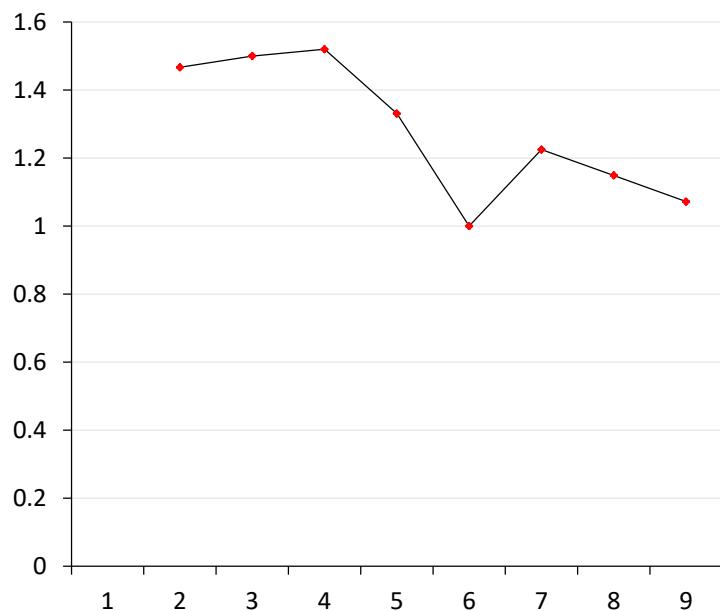
$x=5$

b	$r_u(z=1)$	$r_{max}(z=n)$	$q = r_{max}/r_u$
2	6.658211483	9.816536875	1.474 ($z=4,7,10$)
3	2.261859507	3.596298741	1.590 ($z=5$)
4	1.729715809	2.479901726	1.434 ($z=7$)
5	1.430676558	1.430676559	1.000 ($z=n$)
6	0.898244402	1.226283234	1.365 ($z=6$)
7	0.827087475	1.053372979	1.274 ($z=5$)
8	0.773976032	0.909980382	1.176 ($z=4$)
9	0.732486760	0.787194347	1.075 ($z=10$)



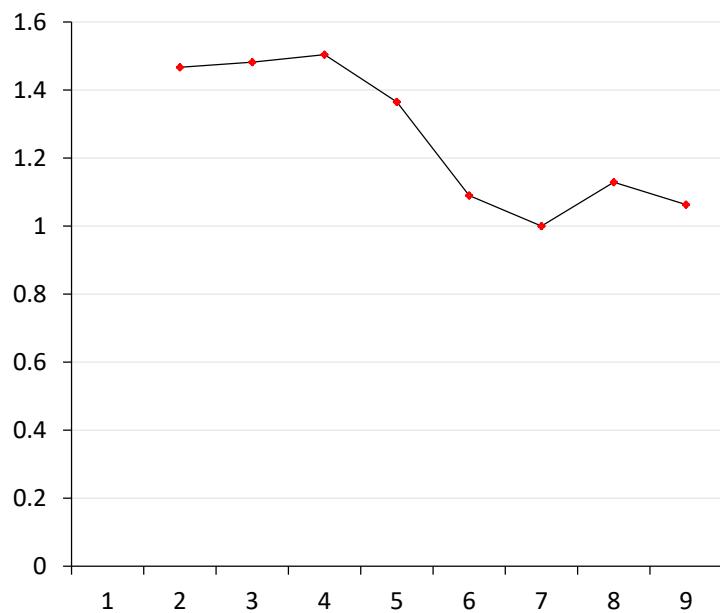
$x=6$

b	$r_u (z=1)$	$r_{max} (z=n)$	$q = r_{max}/r_u$
2	6.781359714	9.950129866	1.467 ($z=7$)
3	2.726833028	4.089497611	1.500 ($z=8$)
4	1.792481250	2.724583829	1.520 ($z=7$)
5	1.489896102	1.983436513	1.331 ($z=9$)
6	1.285097209	1.285097209	1.000 ($z=n$)
7	0.920782221	1.128308862	1.225 ($z=8$)
8	0.861654167	0.989867762	1.149 ($z=7$)
9	0.815464877	0.874001168	1.072 ($z=5$)



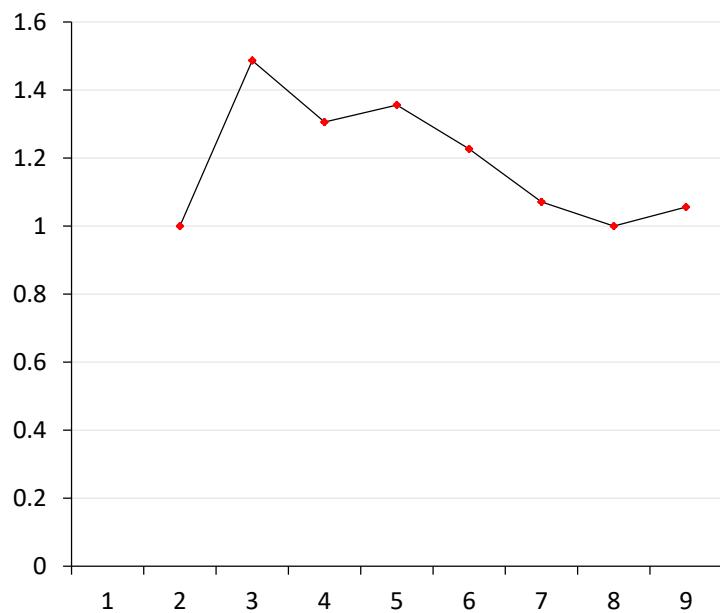
$x=7$

b	$r_u (z=1)$	$r_{max} (z=n)$	$q = r_{max}/r_u$
2	6.794415866	9.964328045	1.467 ($z=5,10$)
3	2.771243749	4.106278008	1.482 ($z=3,4,7,8,10$)
4	1.850219859	2.783500206	1.504 ($z=5,10$)
5	1.543959311	2.107903946	1.365 ($z=5$)
6	1.338290833	1.459139913	1.090 ($z=5$)
7	1.183294662	1.183294663	1.000 ($z=n$)
8	0.935784974	1.056607345	1.129 ($z=7,9$)
9	0.885621875	0.941726356	1.063 ($z=8$)



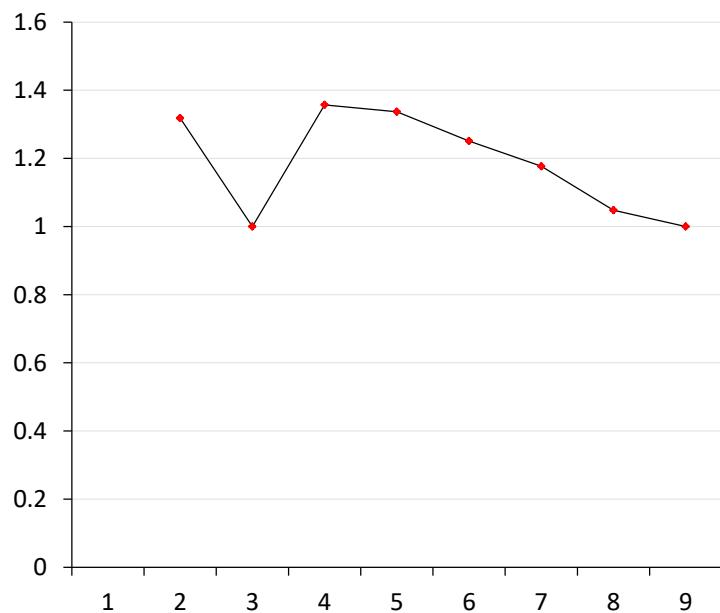
$x=8$

b	$r_u (z=1)$	$r_{max} (z=n)$	$q = r_{max}/r_u$
2	9.965784285	9.965784285	1.000 ($z=n$)
3	2.813588092	4.182651266	1.487 ($z=6,8,9$)
4	2.160964047	2.821928095	1.306 ($z=2,4,6,8,10$)
5	1.593692641	2.160256515	1.356 ($z=7$)
6	1.386852807	1.701032115	1.227 ($z=7$)
7	1.232274406	1.319227972	1.071 ($z=6$)
8	1.107309365	1.107309365	1.000 ($z=n$)
9	0.946394630	0.999837307	1.056 ($z=6,8,9$)



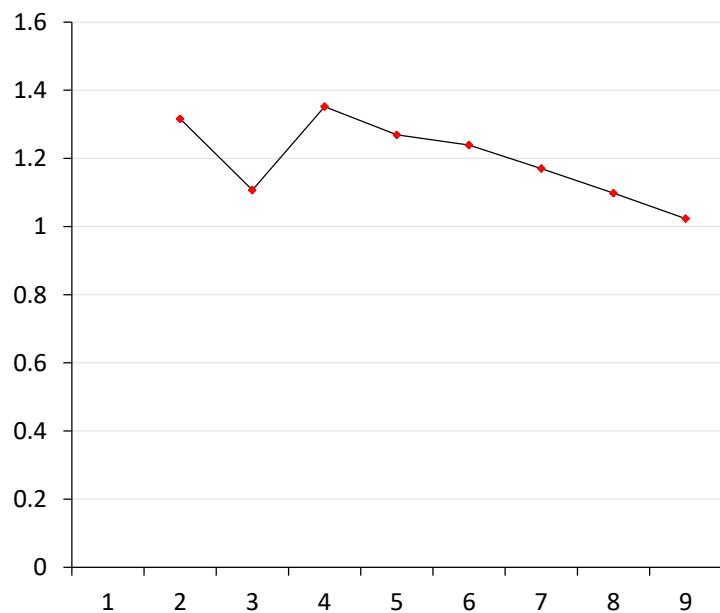
$x=9$

b	$r_u (z=1)$	$r_{max} (z=n)$	$q = r_{max}/r_u$
2	9.967226259	13.137138552	1.318 ($z=6$)
3	4.191806549	4.191806549	1.000 ($z=n$)
4	2.196158711	2.980586953	1.357 ($z=6$)
5	1.639738513	2.193061753	1.337 ($z=6$)
6	1.431525493	1.790389951	1.251 ($z=9$)
7	1.276989408	1.502419186	1.177 ($z=8$)
8	1.153143873	1.208720287	1.048 ($z=6$)
9	1.047951637	1.047951638	1.000 ($z=n$)



$x = 10$

b	$r_u(z=1)$	$r_{max}(z=n)$	$q = r_{max}/r_u$
2	9.980139578	13.138464968	1.316 ($z=4,7,10$)
3	4.200863730	4.648524836	1.107 ($z=7$)
4	2.229715809	3.014650569	1.352 ($z=4,7,10$)
5	1.861353116	2.362122701	1.269 ($z=7$)
6	1.472885940	1.824276797	1.239 ($z=4$)
7	1.318123223	1.541601527	1.170 ($z=6$)
8	1.194987500	1.311822583	1.098 ($z=10$)
9	1.091329169	1.116399340	1.023 ($z=7$)



Graph to show combined distributions of q

